# Math 4300
# Homework 2
# Solutions

$\boxed{① (a)}$ $f: L_2 \to \mathbb{R}$ given by $f(a,y) = y$ is the standard ruler

$f(-2,-3) = -3$

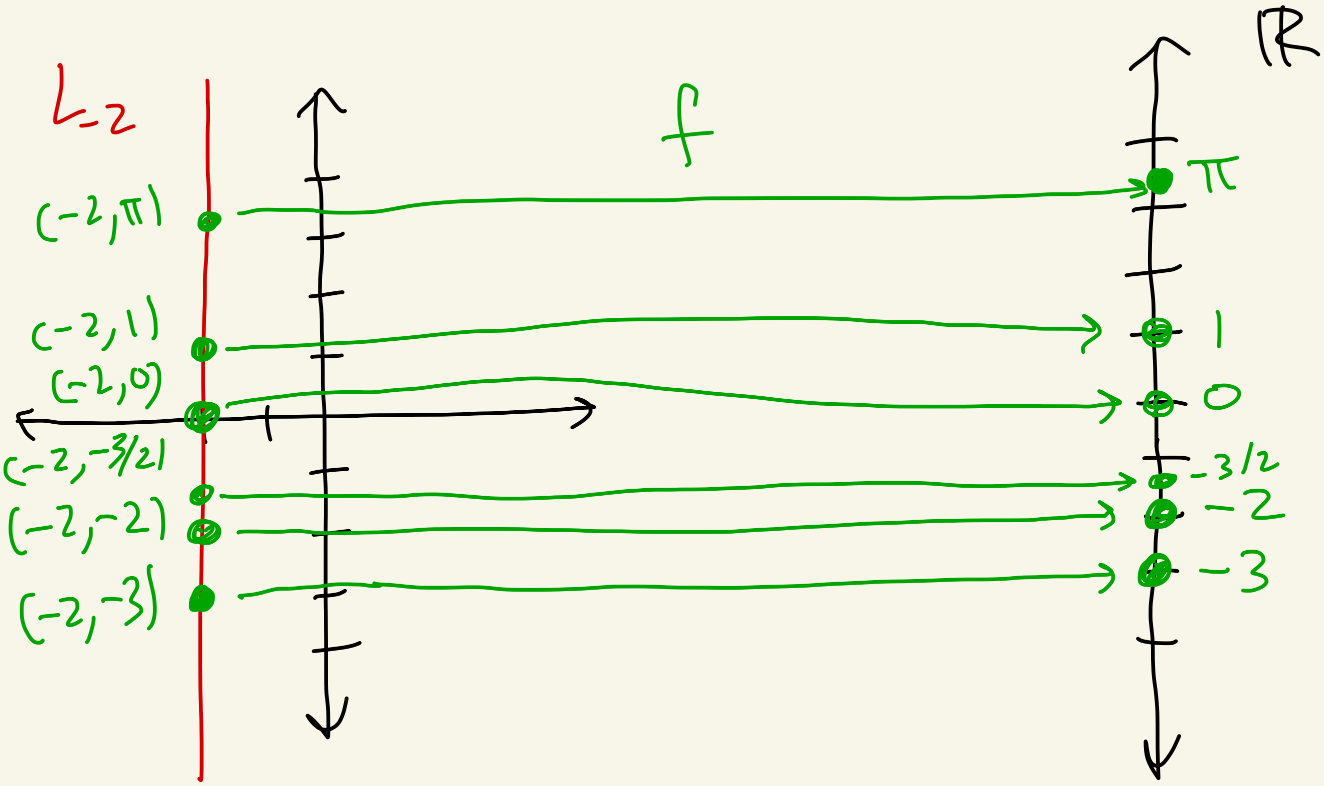
$f(-2,-2) = -2$

$f(-2, -3/2) = -3/2$

$f(-2,0) = 0$

$f(-2,1) = 1$

$f(-2,\pi) = \pi$



$\boxed{①\ (b)}$ The standard ruler for $L_{m,b} = L_{-2,4}$ is $f : L_{-2,4} \longrightarrow \mathbb{R}$

where $f(x, mx+b) = x\sqrt{1+m^2} = \sqrt{5}\, x$

$f(-2,8) = -2\sqrt{5} \approx -4.472$

$f(-1,6) = -\sqrt{5} \approx -2.236$

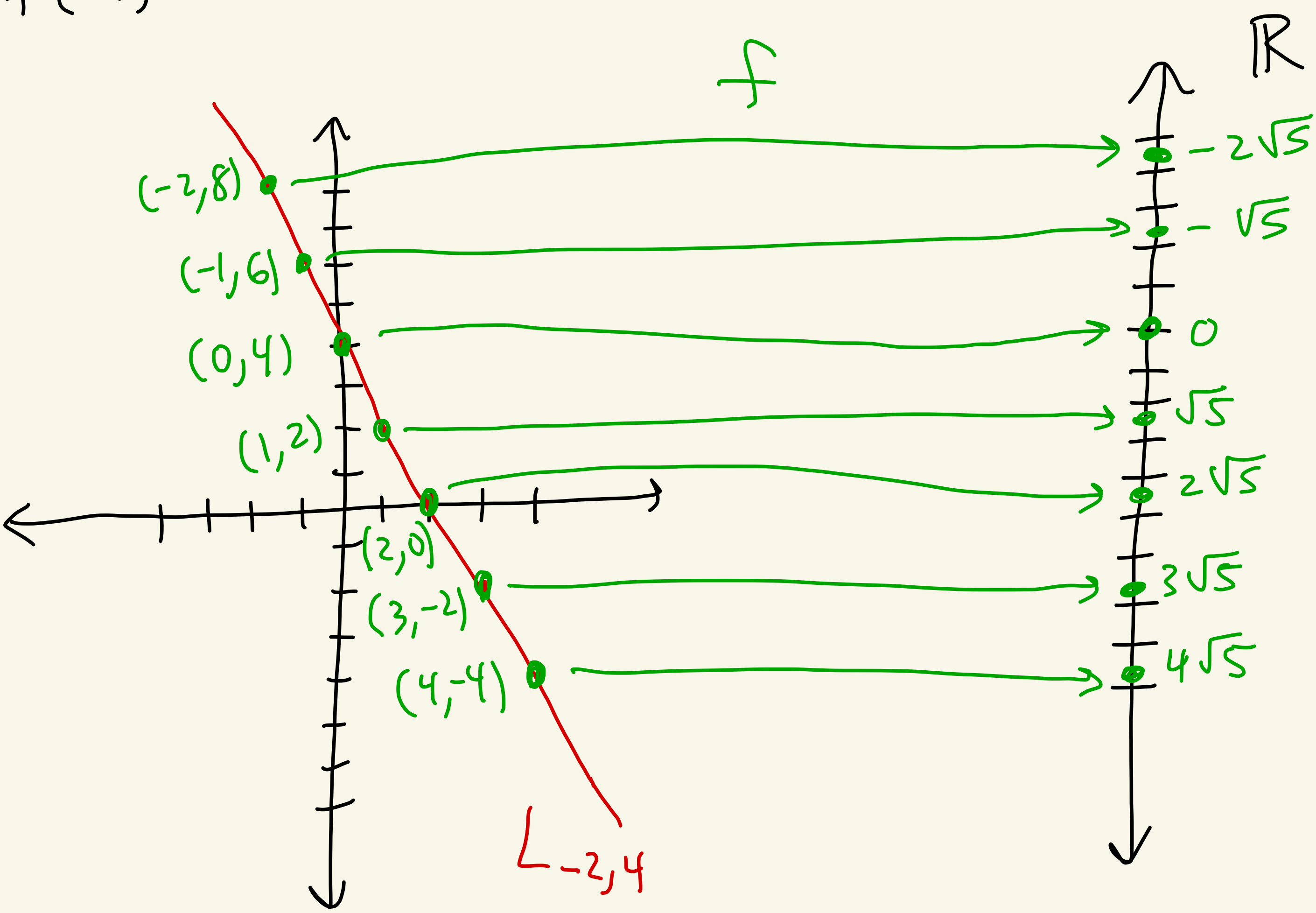
$f(0,4) = 0\sqrt{5} = 0$

$f(1,2) = \sqrt{5} \approx 2.236$

$f(2,0) = 2\sqrt{5} \approx 4.472$

$f(3,-2) = 3\sqrt{5} \approx 6.708$

$f(4,-4) = 4\sqrt{5} \approx 8.944$



**2 (a)** The standard ruler on ${}_2L$ is $f : {}_2L \to \mathbb{R}$ where $f(a,y) = \ln(y)$

$$f(2, 0.001) = \ln(0.001) \approx -9.21$$

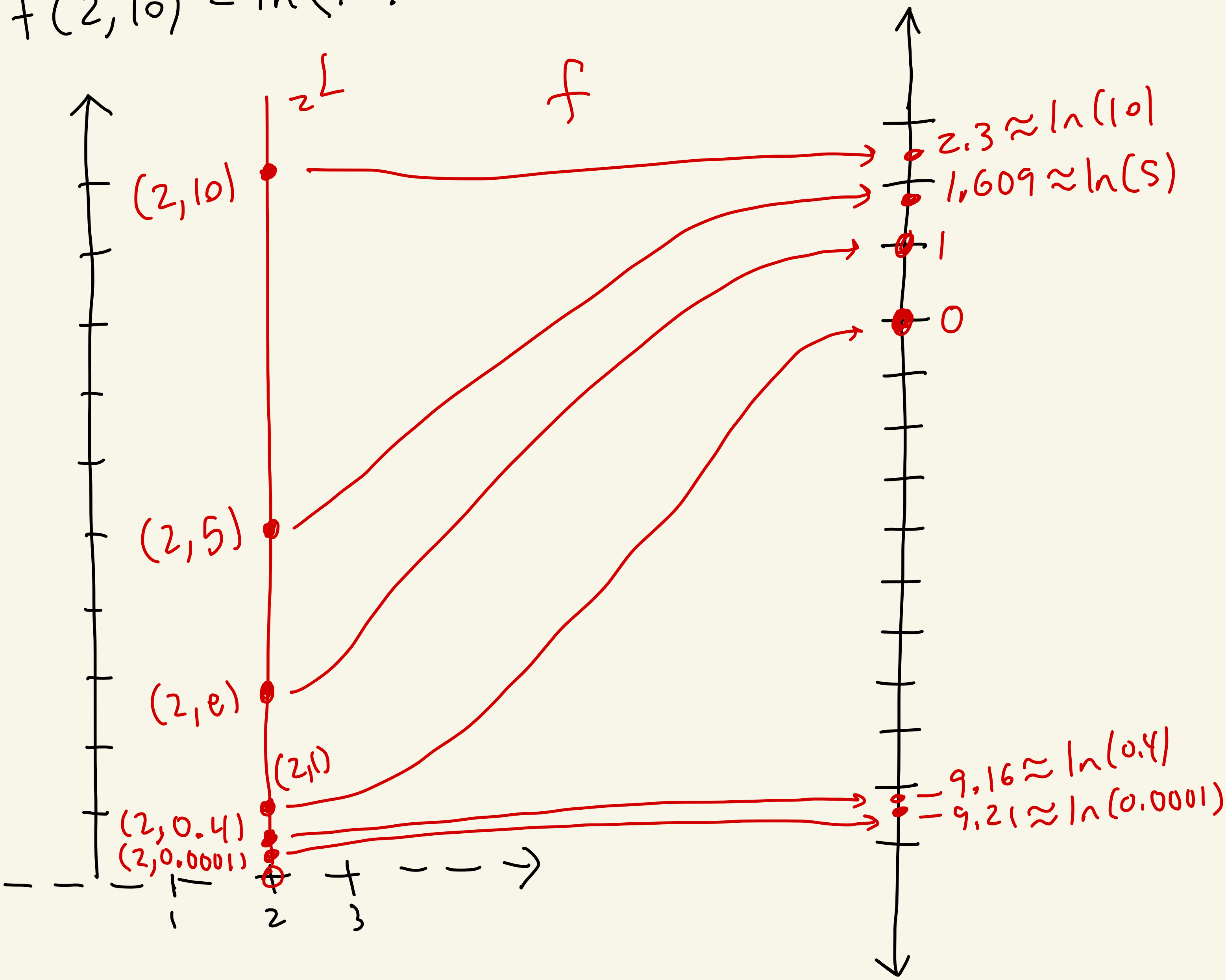
$$f(2, 0.4) = \ln(0.4) \approx -0.916$$

$$f(2,1) = \ln(1) = 0$$

$$f(2,e) = \ln(e) = 1$$

$$f(2,5) = \ln(5) \approx 1.609$$

$$f(2,10) = \ln(10) \approx 2.3$$



② (b) The standard ruler is

$$f : {}_1L_{\sqrt{10}} \to \mathbb{R} \quad \text{where} \quad f(x,y) = \ln\left(\frac{x-1+\sqrt{10}}{y}\right)$$

$$f(-2.16, 0.12) = \ln\left(\frac{-2.16-1+\sqrt{10}}{0.12}\right) \approx \ln(0.01898) \approx -3.964$$

$$f(-1, \sqrt{6}) = \ln\left(\frac{-1-1+\sqrt{10}}{\sqrt{6}}\right) \approx \ln(0.4745) \approx -0.745$$

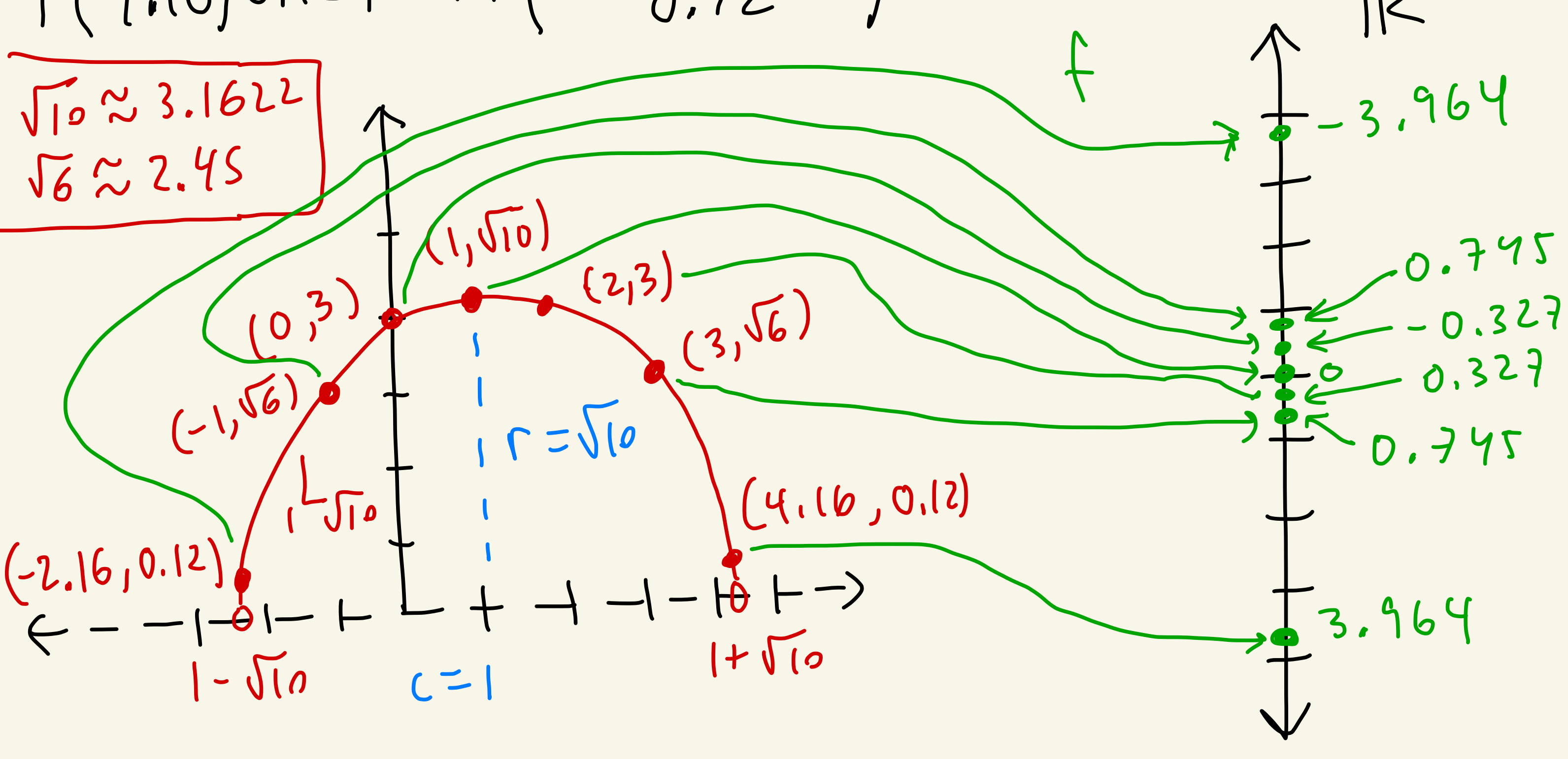
$$f(0,3) = \ln\left(\frac{0-1+\sqrt{10}}{3}\right) \approx \ln(0.72) \approx -0.327$$

$$f(1, \sqrt{10}) = \ln\left(\frac{1-1+\sqrt{10}}{\sqrt{10}}\right) = \ln(1) = 0$$

$$f(2,3) = \ln\left(\frac{2-1+\sqrt{10}}{3}\right) \approx \ln(1.387) \approx 0.327$$

$$f(3, \sqrt{6}) = \ln\left(\frac{3-1+\sqrt{10}}{\sqrt{6}}\right) \approx \ln(2.10749) \approx 0.745$$

$$f(4.16, 0.12) = \ln\left(\frac{4.16-1+\sqrt{10}}{0.12}\right) \approx \ln(52.686) \approx 3.964$$



③

(a) $d((1,2),(3,4)) = \sqrt{(1-3)^2 + (2-4)^2}$

$= \sqrt{4+4}$

$= \sqrt{8}$

(b) $d((-3,1),(5,10)) = \sqrt{(-3-5)^2 + (1-10)^2}$

$= \sqrt{64+81}$

$= \sqrt{145}$

$\boxed{④\ (a)}$

$P=(1,2)$ and $Q=(5,6)$

We need to know what line $P$ and $Q$ live on.

They have different $x$-coordinates so it's not a vertical line. It must be some line ${}_cL_r$.

Let's find it.

Plug $P$ and $Q$ into $(x-c)^2+y^2=r^2$ to get

$$\boxed{\begin{aligned}(1-c)^2+2^2&=r^2\\(5-c)^2+6^2&=r^2\end{aligned}}$$

$\leftarrow$ $\boxed{\text{plug in } P}$

$\leftarrow$ $\boxed{\text{plug in } Q}$

This gives

$$\boxed{\begin{aligned}1-2c+c^2+4&=r^2\\25-10c+c^2+36&=r^2\end{aligned}}$$

$$\Downarrow$$

$$\boxed{\begin{aligned}-2c+c^2+5&=r^2 \quad ①\\-10c+c^2+61&=r^2 \quad ②\end{aligned}}$$

① − ② gives $8c - 56 = 0$.

So, $c = \frac{56}{8} = 7$.

And ① then gives $r = \sqrt{\underbrace{4+(1-7)^2}_{4+(1-c)^2}} = \sqrt{40}$

$= 2\sqrt{10}$

$\approx 6.32$

So, $P=(1,2)$, $Q=(5,6)$ lie on the hyperbolic line ${}_7L_{2\sqrt{10}}$

Thus,

$$d_H(P,Q) = \left| \ln\left( \frac{\frac{1-7+2\sqrt{10}}{2}}{\frac{5-7+2\sqrt{10}}{6}} \right) \right| = \left| \ln\left( \frac{\frac{-6+2\sqrt{10}}{2}}{\frac{-2+2\sqrt{10}}{6}} \right) \right|$$

$$\approx \left| \ln\left( \frac{0.16227766}{0.72075922} \right) \right|$$

$$\approx |-1.49|$$

$$= 1.49$$

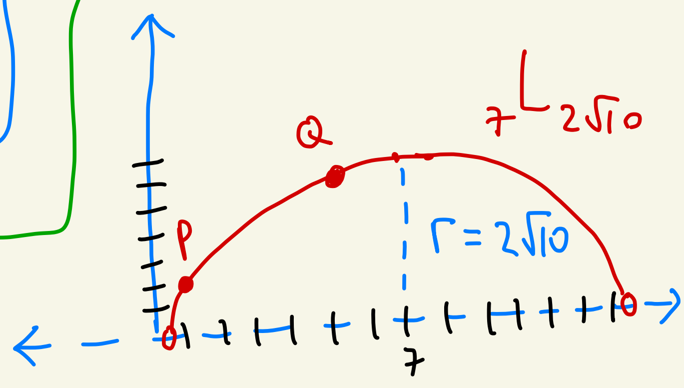
$P=(x_1,y_1)=(1,2)$

$Q=(x_2,y_2)=(5,6)$

$$d_H(P,Q) = \left| \ln\left( \frac{\frac{x_1-c+r}{y_1}}{\frac{x_2-c+r}{y_2}} \right) \right|$$

Thus, $\boxed{d_H(P,Q) \approx 1.49}$

PICTURE



$\boxed{4 (b)}$ $P = (6, \pi^2)$, $Q = (6, 2)$

P and Q lie on the vertical line ${}_6L$

Thus,

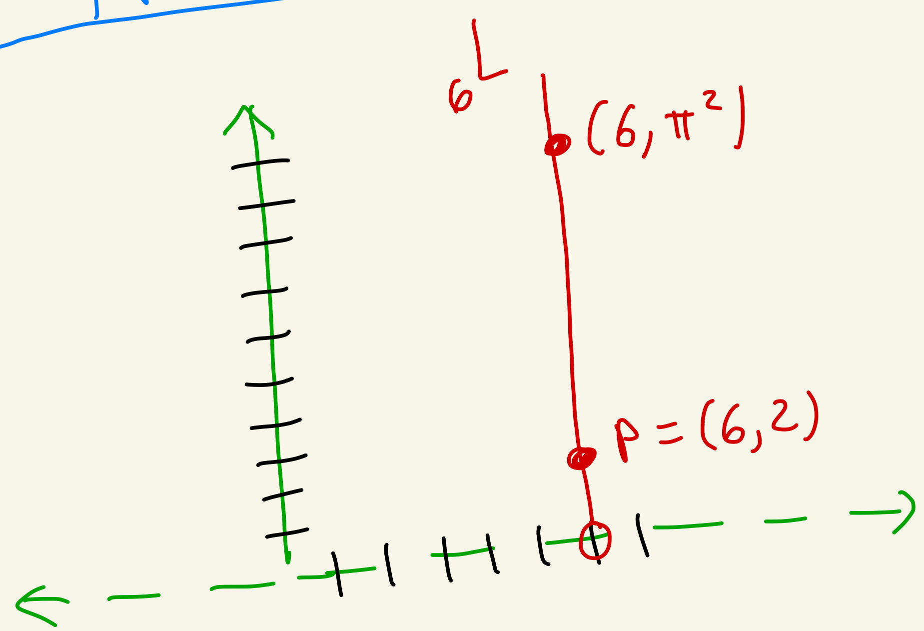
$$d_H(P, Q) = \left| \ln\left(\frac{\pi^2}{2}\right) \right| \approx |1.59631|$$

$$= 1.59631$$

$$d_H((a, y_1), (a, y_2)) = \left| \ln\left(\frac{y_1}{y_2}\right) \right|$$

when both points lie on ${}_aL$

PICTURE



$\boxed{5}$ The standard ruler for $L_{3,-3}$ is

$$f : L_{3,-3} \to \mathbb{R} \text{ where } f(x, 3x-3) = x\sqrt{1+3^2} = \sqrt{10}\,x$$

The problem is to find $P = (x_1, y_1)$ where $f(x_1, y_1) = -2$.

We need to solve $\sqrt{10}\,x_1 = -2$.

Thus, $x_1 = \frac{-2}{\sqrt{10}}$

To find $y_1$ we plug $(x_1, y_1) = \left(\frac{-2}{\sqrt{10}}, y_1\right)$ into the line $L_{3,-3}$ $\leftarrow$ $\boxed{y = 3x - 3}$

This gives $y_1 = \underbrace{3\left(\frac{-2}{\sqrt{10}}\right) - 3}_{3x_1 - 3} = \frac{-6}{\sqrt{10}} - 3$

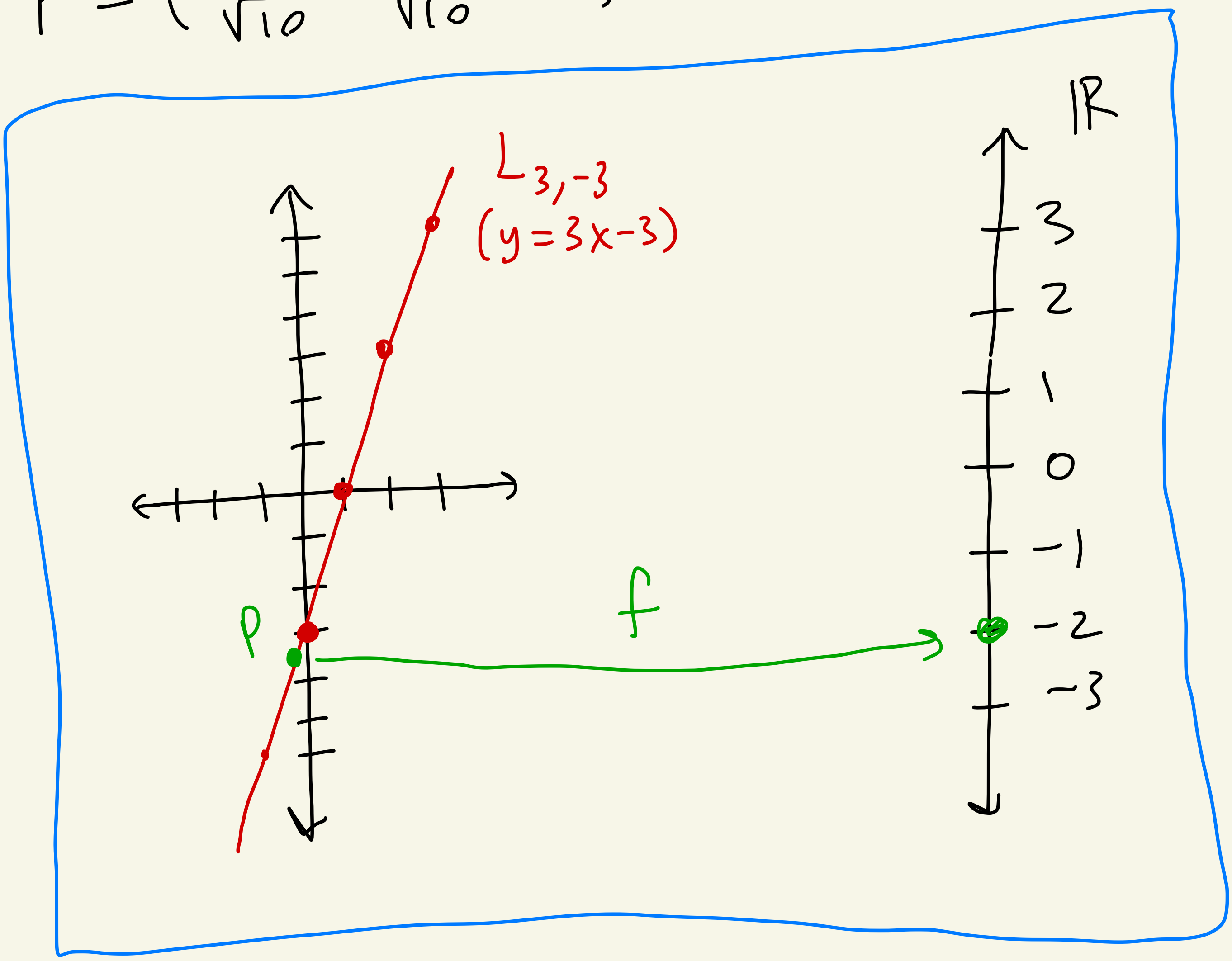
Answer

Thus, $P = \left(-\frac{2}{\sqrt{10}}, \frac{-6}{\sqrt{10}} - 3\right)$ has coordinate $-2$ using the standard ruler.

PICTURE ON NEXT PAGE

Let's see a picture to understand this answer more.

$$p = \left(\frac{-2}{\sqrt{10}}, \frac{-6}{\sqrt{10}} - 3\right) \approx (-0{,}632, -4{,}897)$$



6 (a) Let $P=(2,3)$, $Q=(2,5)$.

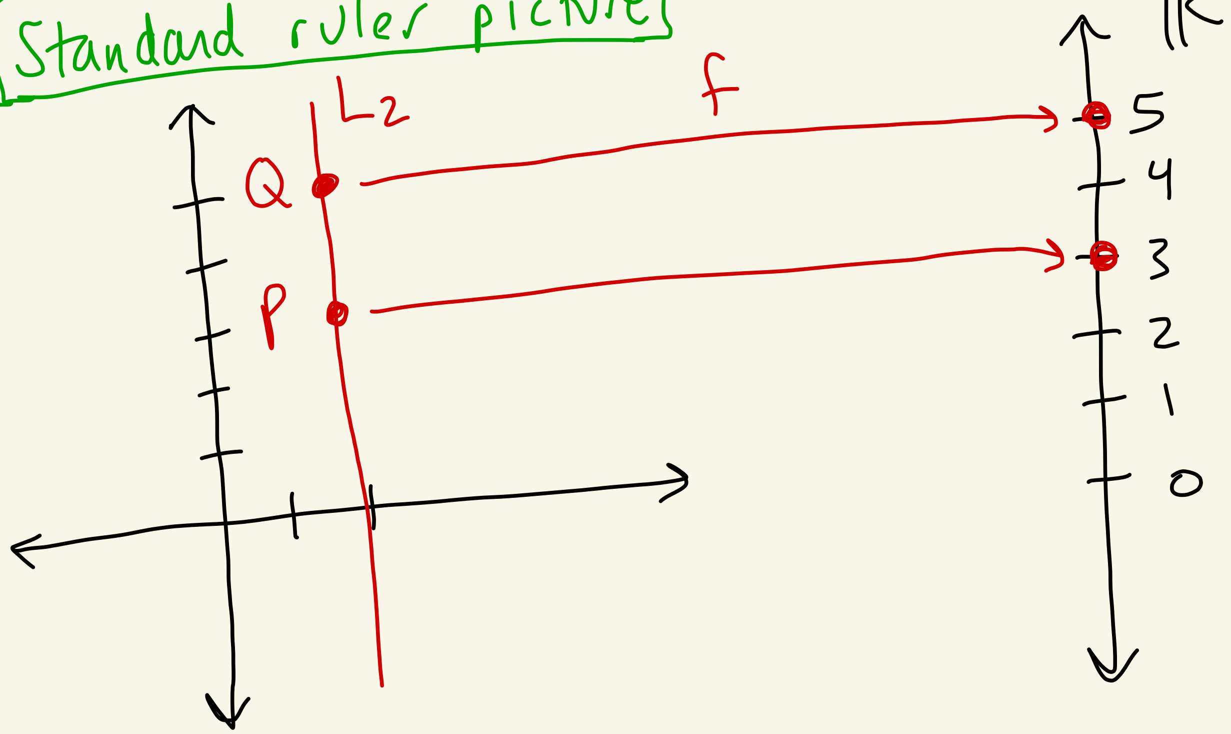
Then $\overleftrightarrow{PQ} = L_2$

The standard ruler on $\overleftrightarrow{PQ} = L_2$ is given by

$f: L_2 \longrightarrow \mathbb{R}$ where $f(a,y)=y$

$f(2,3)=3$
$f(2,5)=5$

Standard ruler picture



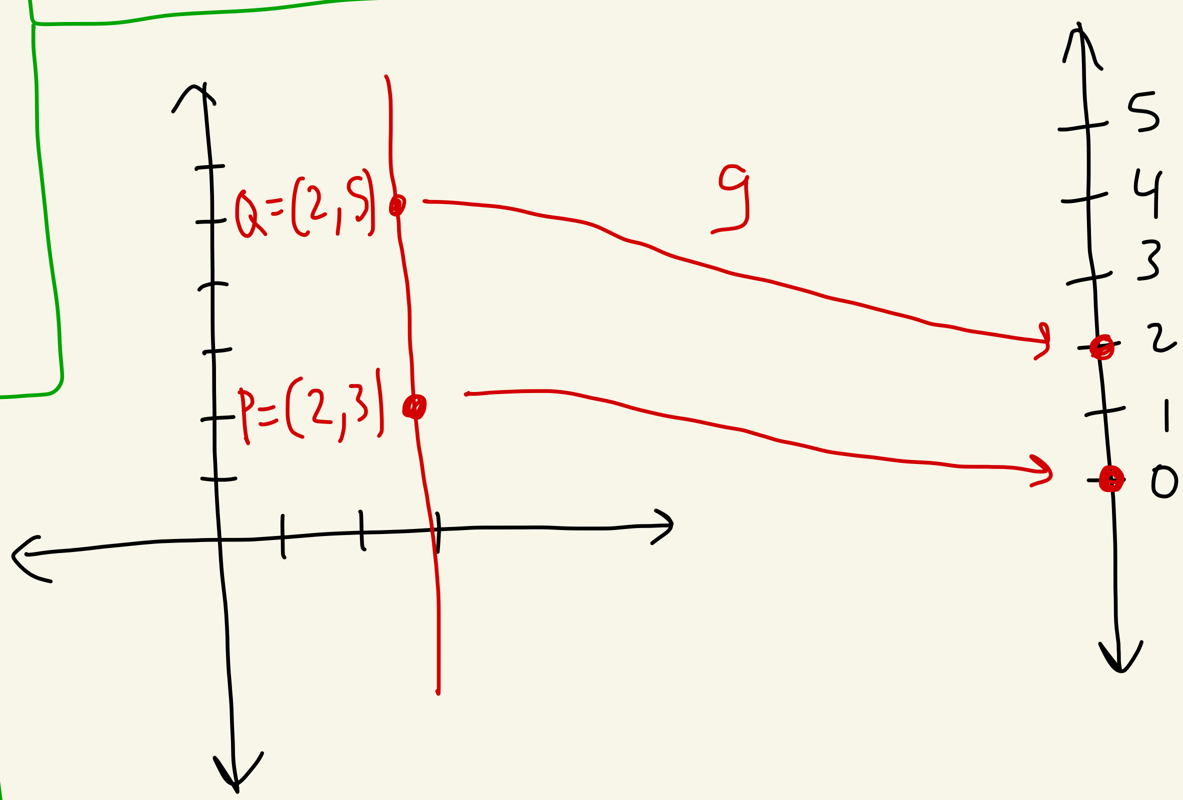

Since $f(Q)=5>3=f(P)$ we need to shift this ruler by $f(P)=3$.

The new ruler is $g : L_2 \to \mathbb{R}$

given by
$$\boxed{\begin{aligned} g(a,y) &= f(a,y) - f(P) \\ &= f(a,y) - 3 \\ &= y - 3 \end{aligned}}$$

Here $g(P) = g(2,3) = 3 - 3 = 0$
$g(Q) = g(2,5) = 5 - 3 = 2 > 0.$

Picture of $g$



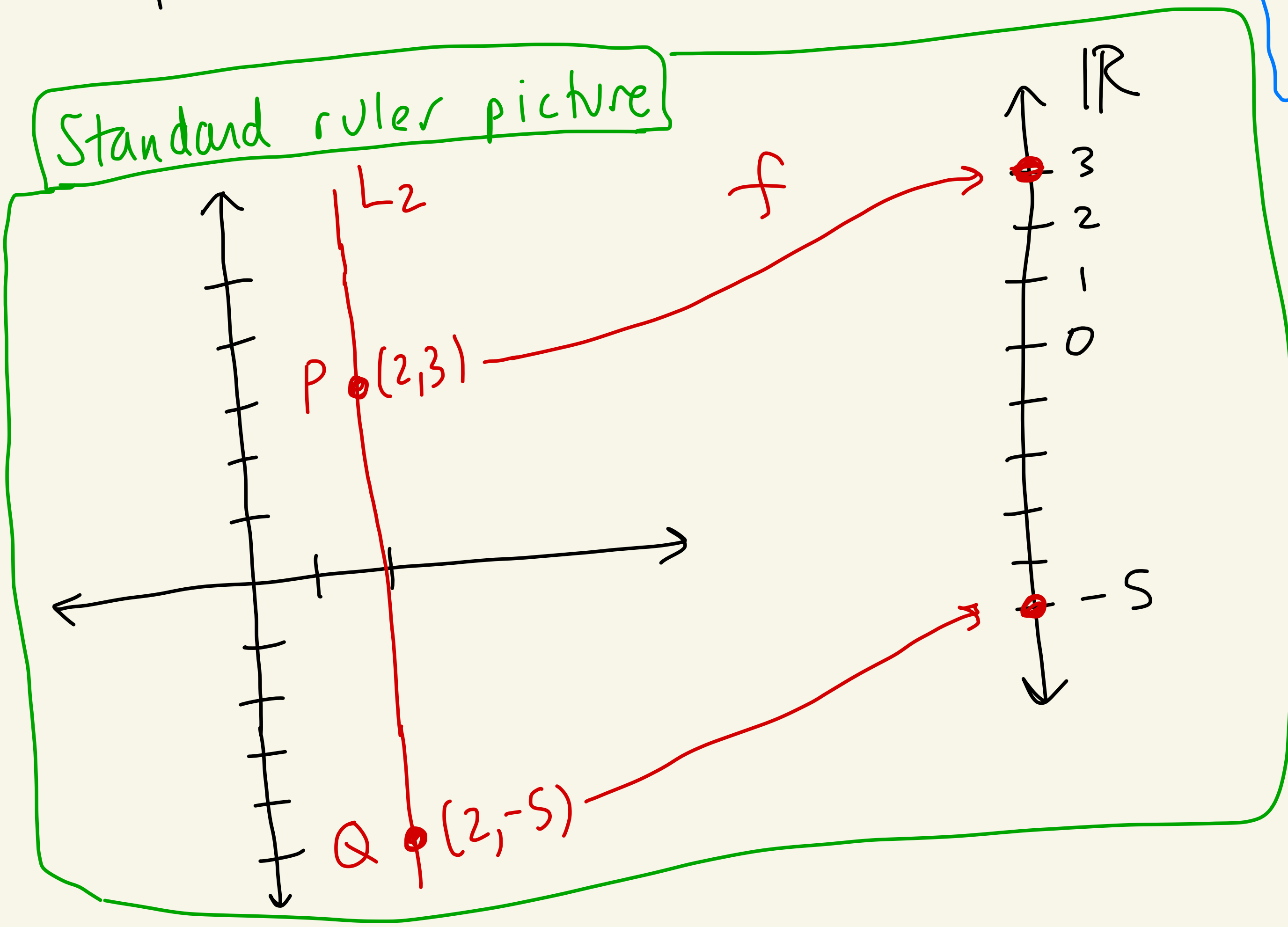
6 (b) Let $P=(2,3)$, $Q=(2,-5)$.

Then $\overleftrightarrow{PQ} = L_2$

The standard ruler on $\overleftrightarrow{PQ} = L_2$ is given by

$f : L_2 \to \mathbb{R}$ where $f(a,y) = y$

$f(2,3)=3$
$f(2,-5)=-5$




Since $f(Q) = -5 < 3 = f(P)$ we need to shift this ruler by $f(P)=3$.
and then multiply by $-1$.

The new ruler is $g : L_2 \to \mathbb{R}$

given by $g(a,y) = -(f(a,y) - f(P))$

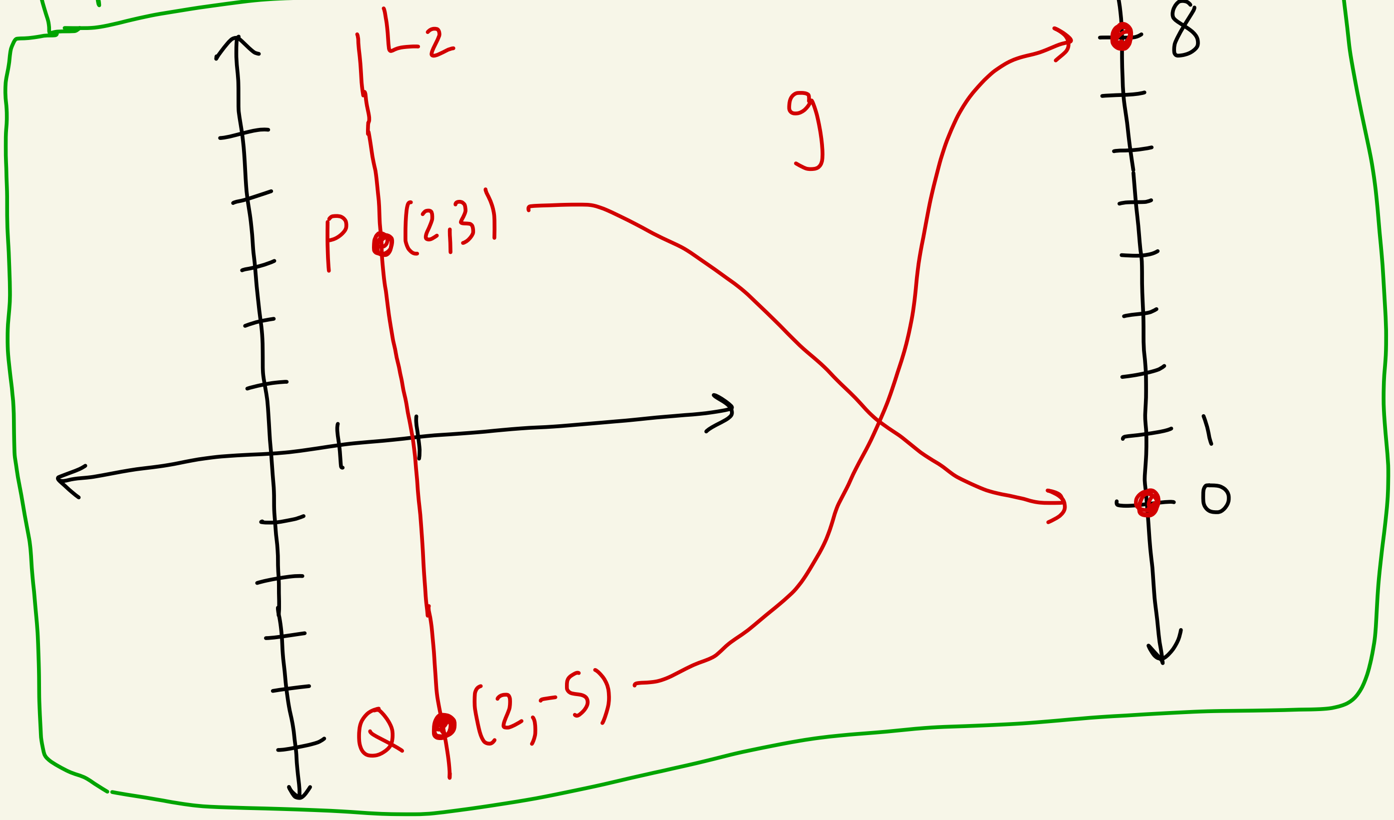
$= -(f(a,y) - 3)$

$= -y + 3$

Here $g(P) = g(2,3) = -3 + 3 = 0$

$g(Q) = g(2,5) = -(-5) + 3 = 8$

picture of $g$



$\boxed{⑥(c)}$ $P=(2,3)$, $Q=(4,0)$ do not lie on a vertical line.

Let $m = \dfrac{0-3}{4-2} = \dfrac{-3}{2}$.

What is $b$?

Plug $P=(2,3)$ into $y = \frac{-3}{2}x + b$ to get

$3 = \left(-\frac{3}{2}\right)(2) + b$. This gives $b = 6$.

Thus, $P$ and $Q$ lie on $L_{-\frac{3}{2},6}$.

The standard ruler is $f: L_{-3/2,6} \to \mathbb{R}$

where $f(x,y) = x\sqrt{1+\left(-\frac{3}{2}\right)^2} = x\sqrt{13/4}$

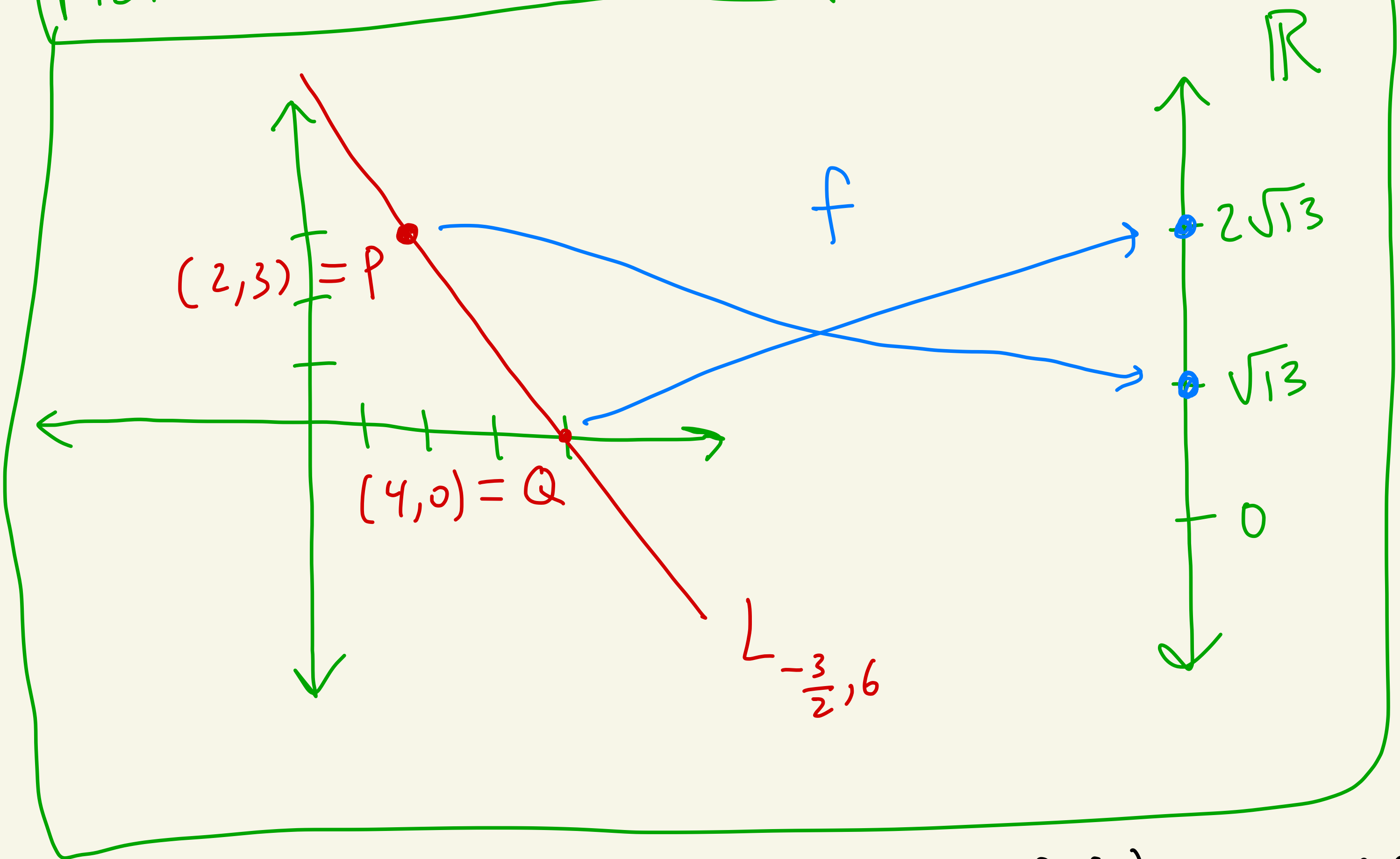
$$= \frac{\sqrt{13}}{2}x$$

Here we have

$f(P) = f(2,3) = \frac{\sqrt{13}}{2}\cdot 2 = \sqrt{13} \approx 3.6$

$f(Q) = f(4,0) = \frac{\sqrt{13}}{2}\cdot 4 = 2\sqrt{13} \approx 7.2$

picture of standard ruler




Since $f(Q)=2\sqrt{13}>\sqrt{13}=f(P)$ we need to shift $f$ by $f(P)=\sqrt{13}$

Set $g: L_{-\frac{3}{2},6} \to \mathbb{R}$ where

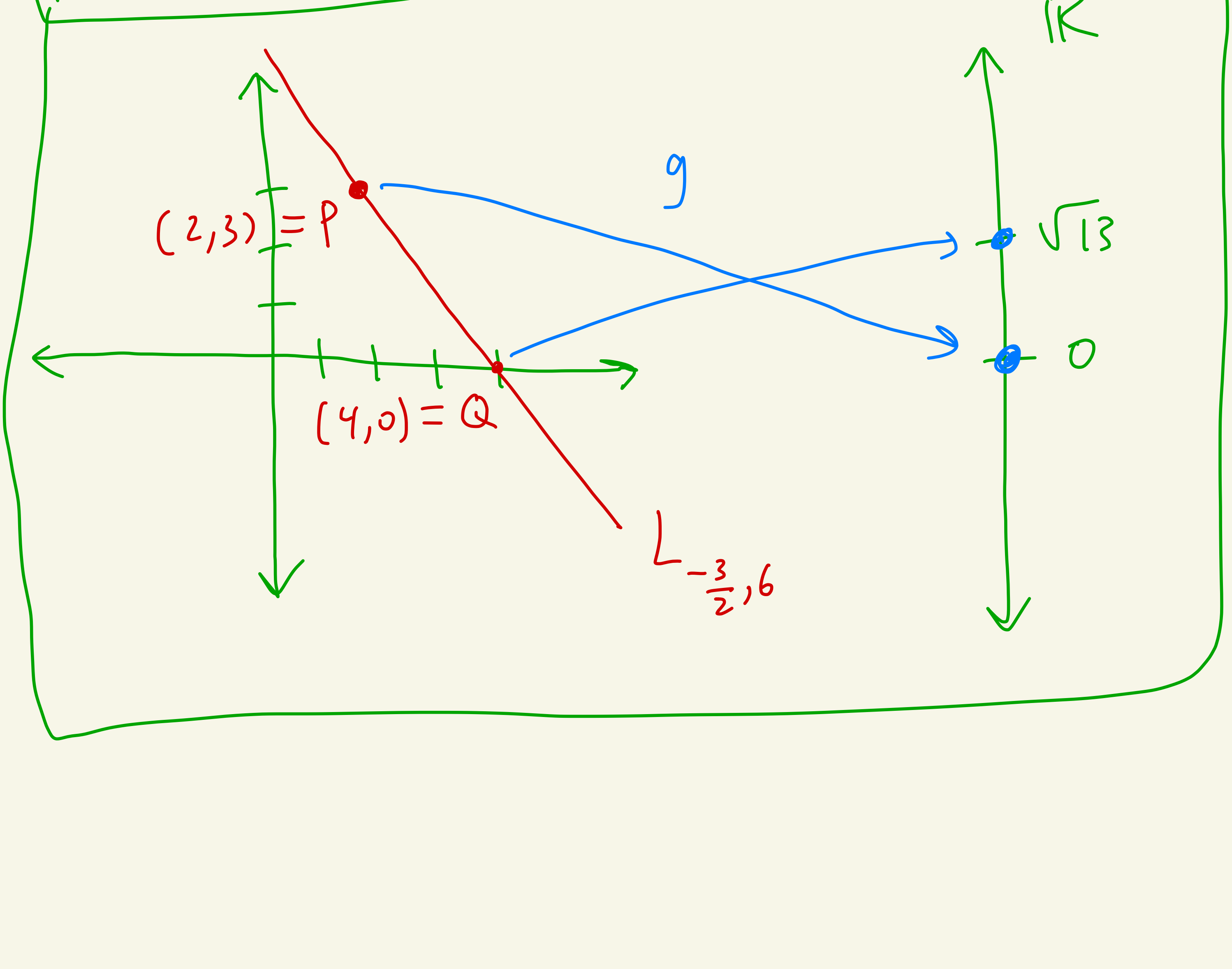
$$g(x,y) = f(x,y) - f(P) = \frac{\sqrt{13}}{2}x - \sqrt{13}$$

Then, $g(P) = g(2,3) = \frac{\sqrt{13}}{2} \cdot 2 - \sqrt{13} = 0$

$$g(Q) = g(4,0) = \frac{\sqrt{13}}{2} \cdot 4 - \sqrt{13} = \sqrt{13} > 0$$

So, $g$ is the ruler that we want.

picture of $g$

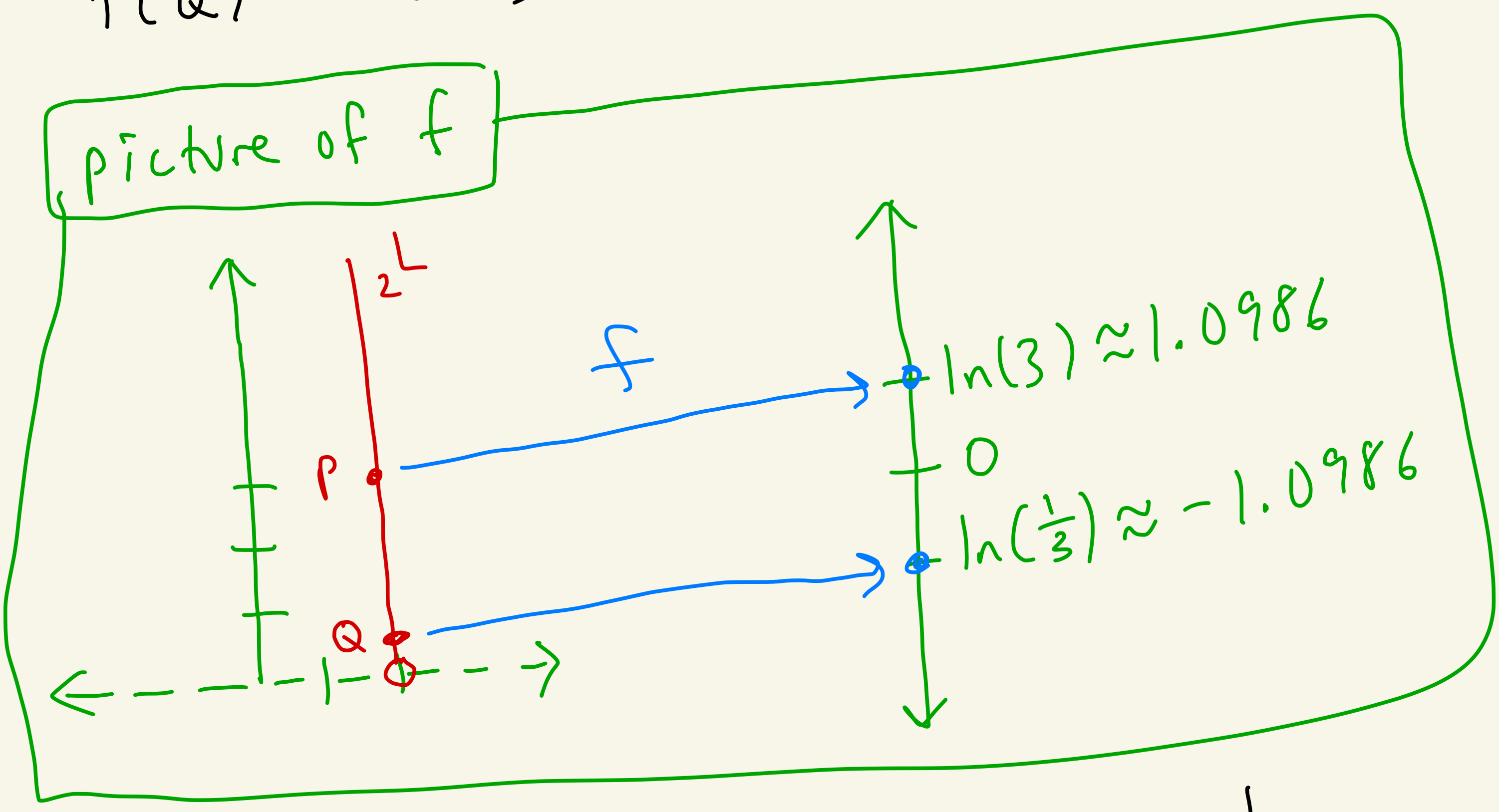


**7** (a) $P=(2,3)$, $Q=(2,\frac{1}{3})$ lie on the line ${}_2L$. The standard ruler is $f: {}_2L \to \mathbb{R}$ where $f(a,y)=\ln(y)$

We have

$f(P)=f(2,3)=\ln(3)\approx 1.0986$

$f(Q)=f(2,\frac{1}{3})=\ln(\frac{1}{3})\approx -1.0986$




Since $f(Q)<f(P)$ we need to shift $f$ by $f(P)$ and multiply by $-1$.

$$\boxed{\begin{aligned} &\text{Set } g : {}_2L \to \mathbb{R} \text{ where} \\ &g(a,y) = -\big(f(a,y) - f(P)\big) \\ &\phantom{g(a,y)} = -\big(\ln(y) - \ln(3)\big) \\ &\phantom{g(a,y)} = -\ln(y) + \ln(3) \end{aligned}}$$

Then, $g(P) = g(2,3) = -\ln(3) + \ln(3) = 0$

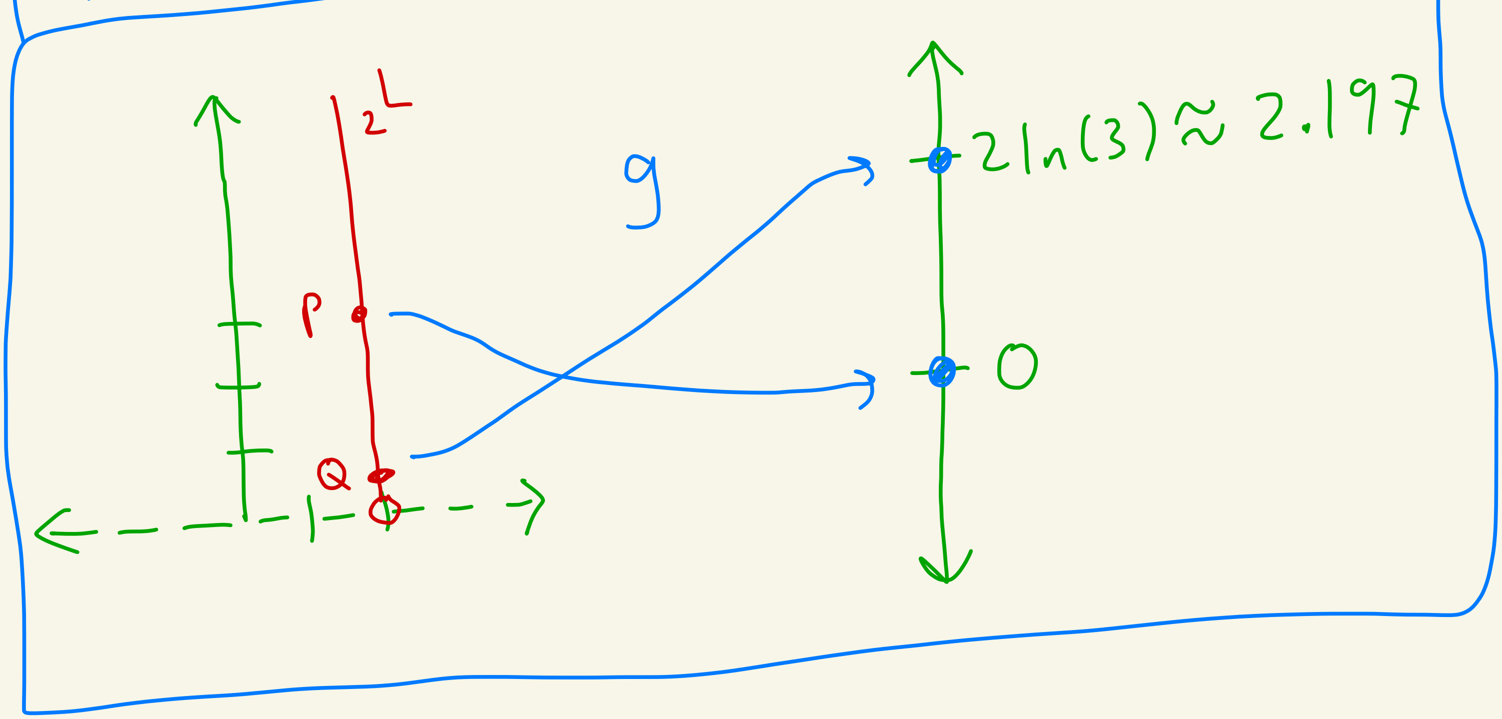
and $g(Q) = g(2, \frac{1}{3}) = -\ln(\frac{1}{3}) + \ln(3)$

$$\overset{-\ln(z) = \ln(z^{-1})}{=} \ln(3) + \ln(3)$$

$$= 2\ln(3) \approx 2.197 > 0$$

So, this $g$ satisfies the conditions.

picture of $g$



$\boxed{(7)(b)}$ $P=(2,3)$, $Q=(-1,6)$

They do not lie on a vertical line and so $\overleftrightarrow{PQ} = {}_cL_r$ for some $c, r$.

Plug $P$ and $Q$ into $(x-c)^2+y^2=r^2$ to get

$$\boxed{\begin{aligned}(2-c)^2+3^2&=r^2\\(-1-c)^2+6^2&=r^2\end{aligned}}$$

$$\Downarrow$$

$$\boxed{\begin{aligned}-4c+c^2+13&=r^2\\2c+c^2+37&=r^2\end{aligned}} \quad \begin{aligned}&(1)\\&(2)\end{aligned}$$

(1) − (2) gives $-6c-24=0$.

So, $c=-4$

Then, (1) gives $r=\sqrt{(2-(-4))^2+3^2}=\sqrt{45}$
$\approx 6.708$

So, $P=(2,3)$ and $Q=(-1,6)$ lie

on ${}_{-4}L_{\sqrt{45}}$.

The standard ruler is $f: {}_{-4}L_{\sqrt{45}} \to \mathbb{R}$

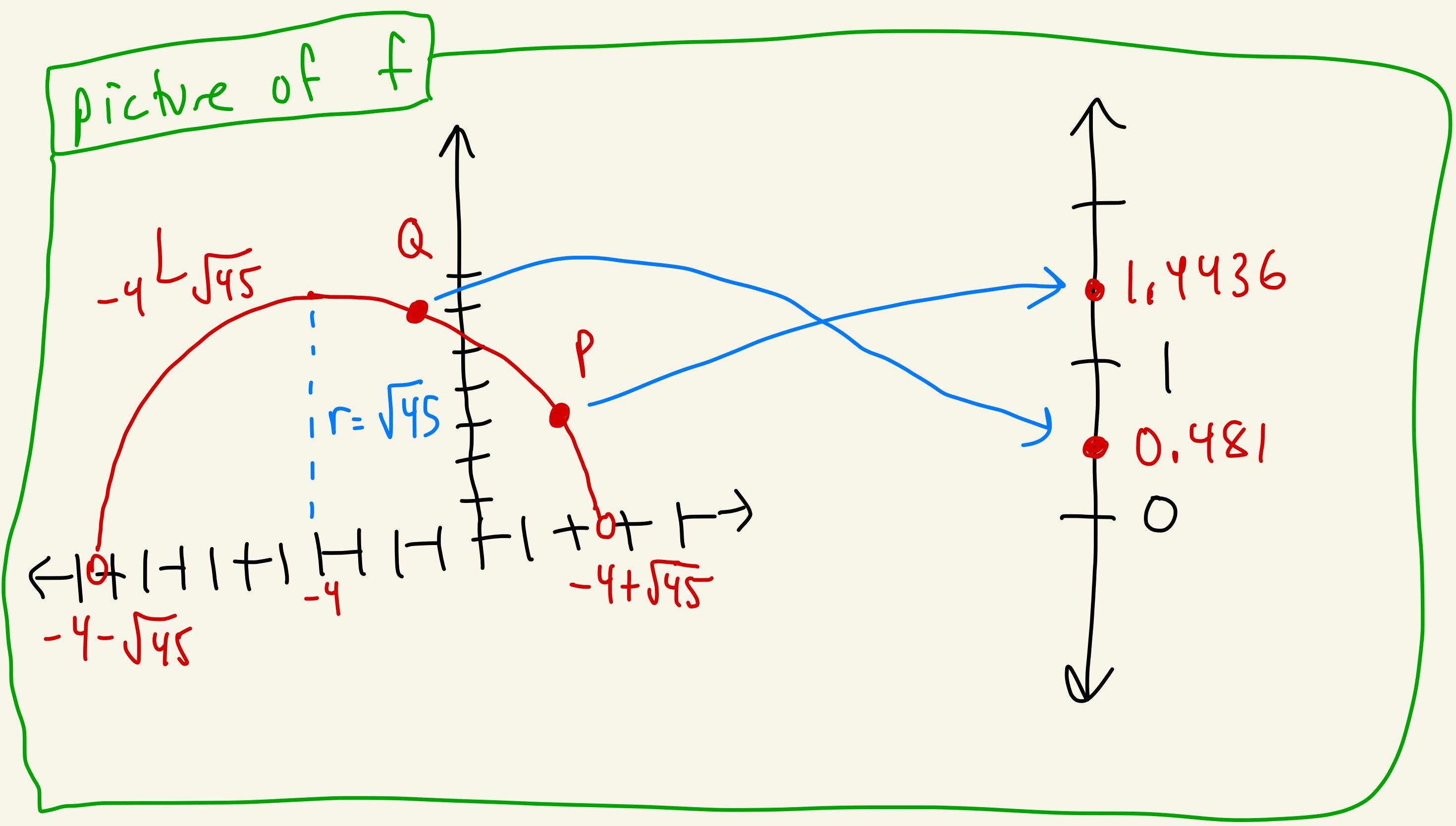
where $f(x,y) = \ln\left(\frac{x-c+r}{y}\right) = \ln\left(\frac{x+4+\sqrt{45}}{y}\right)$

Then,

$$f(P) = f(2,3) = \ln\left(\frac{2+4+\sqrt{45}}{3}\right) \approx \ln(4.236) \approx 1.4436$$

and

$$f(Q) = f(-1,6) = \ln\left(\frac{-1+4+\sqrt{45}}{6}\right) \approx \ln(1.618) \approx 0.481$$




Since $f(Q) < f(P)$ we shift by $f(P)$ and multiply by $-1$.

We get $g: L_{-4\sqrt{45}}$ given by

$$g(x,y) = -\left(f(x,y) - f(P)\right)$$

$$= -\left(\ln\left(\frac{x+4+\sqrt{45}}{y}\right) - \ln\left(\frac{6+\sqrt{45}}{3}\right)\right)$$

$$= -\ln\left(\frac{\left(\frac{x+4+\sqrt{45}}{y}\right)}{\left(\frac{6+\sqrt{45}}{3}\right)}\right)$$

$$\boxed{\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)}$$

$$\boxed{-\ln(c) = \ln(c^{-1})}$$

$$= \ln\left(\frac{\left(\frac{6+\sqrt{45}}{3}\right)}{\left(\frac{x+4+\sqrt{45}}{y}\right)}\right)$$

Then,

$$g(P) = g(2,3) = \ln\left(\frac{\left(\frac{6+\sqrt{45}}{3}\right)}{\left(\frac{2+4+\sqrt{45}}{3}\right)}\right) = \ln(1) = 0$$

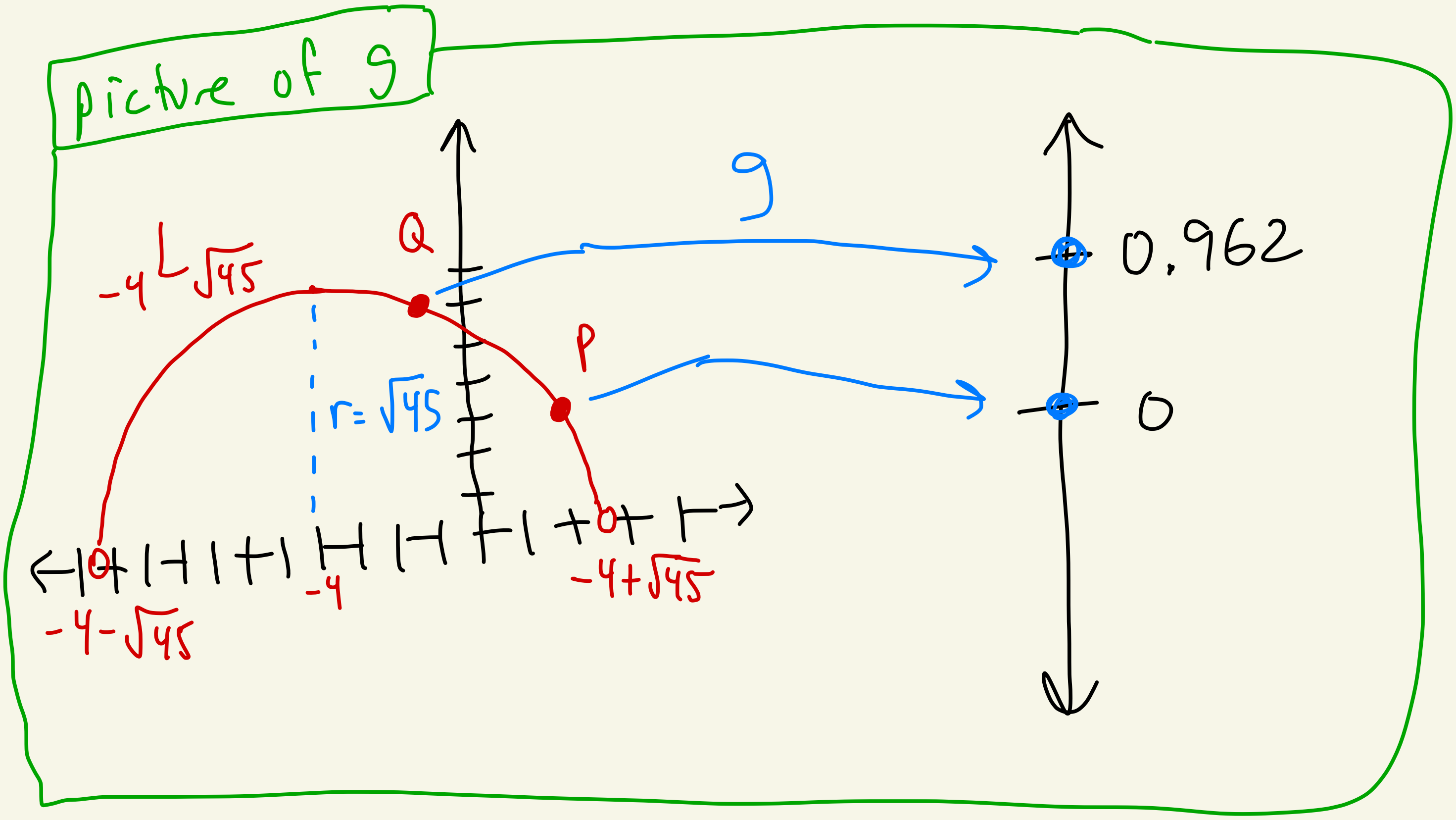
and

$$g(Q) = g(-1,6) = \ln\left(\frac{\left(\frac{6+\sqrt{45}}{3}\right)}{\left(\frac{-1+4+\sqrt{45}}{6}\right)}\right) \approx \ln\left(\frac{4.23607}{1.618}\right)$$

$$\approx 0.962 > 0$$

So, $g(P)=0$ and $g(Q)>0$.

Thus $g$ is the ruler we are looking for.



$\boxed{8}$ Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry.
Let $P$ be a point and $\ell$ be a line where $P$ is on $\ell$.

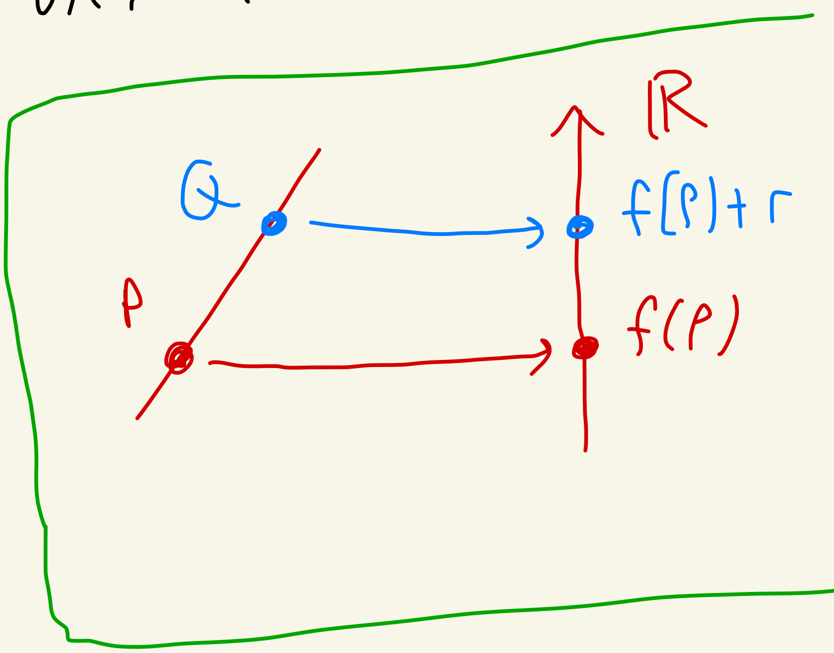
Let $r > 0$.
We must find a point $Q$ on $\ell$ with $d(P,Q) = r$.

Since we are in a metric geometry there exists a ruler $f: \ell \to \mathbb{R}$.

Then $f(P)$ is some real number.

Since $f$ is a bijection, there exists a point $Q \in \ell$ where

$$f(Q) = f(P) + r$$




Then, because $f$ is a ruler we know

$$d(P,Q) = |f(P) - f(Q)|$$

$$= |f(P) - (f(P) + r)|$$

$$= |-r|$$

$$= r$$

since $r > 0$

<u>Note:</u> You could have also picked $Q \in \ell$ where $f(Q) = f(P) - r$ and that would have also worked.

$\blacksquare$

(9) Let $\ell$ be a line in a metric geometry $(\mathscr{P}, \mathscr{L}, d)$. Then there exists a ruler $f: \ell \to \mathbb{R}$. Since $f$ is a bijection and $\mathbb{R}$ is an infinite set, then $\ell$ must also be an infinite set. $\blacksquare$

$\boxed{\text{10 (a)}}$ Let $t \in \mathbb{R}$.

Then,

$$(\cosh(t))^2 - (\sinh(t))^2$$

$$= \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2$$

$$= \frac{(e^{2t} + 2e^t e^{-t} + e^{-2t}) - (e^{2t} - 2e^t e^{-t} + e^{-2t})}{4}$$

$$= \frac{4e^t e^{-t}}{4} = e^0 = 1.$$

$\boxed{\text{10 (b)}}$ Let $t \in \mathbb{R}$. Then $e^t > 0$ and $e^{-t} > 0$.

Thus,

$$\cosh(t) = \frac{e^t + e^{-t}}{2} > \frac{0+0}{2} = 0.$$

$\boxed{10(c)}$ Let $t \in \mathbb{R}$. Then,

$$(\tanh(t))^2 + (\operatorname{sech}(t))^2$$

$$= \frac{(\sinh(t))^2}{(\cosh(t))^2} + \frac{1}{(\cosh(t))^2}$$

$$= \frac{(\sinh(t))^2 + 1}{(\cosh(t))^2} \overset{10(a)}{=} \frac{(\cosh(t))^2}{(\cosh(t))^2}$$

$$= 1$$

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$\boxed{10(d)}$ Let $t \in \mathbb{R}$. From 10(c), we know $\cosh(t) > 0$. Thus,

$$\operatorname{sech}(t) = \frac{1}{\cosh(t)} > 0.$$

(10)(e) One can show that $\tanh(t)$ is always increasing by showing that $(\tanh(t))' > 0$ for all $t$.

We have that

$$(\tanh(t))' = \left(\frac{\sinh(t)}{\cosh(t)}\right)' = \left[\frac{e^t - e^{-t}}{2} \cdot \frac{2}{e^t + e^{-t}}\right]'$$

$$= \left(\frac{e^t - e^{-t}}{e^t + e^{-t}}\right)'$$

$$= \frac{(e^t + e^{-t})(e^t + e^{-t}) - (e^t - e^{-t})(e^t - e^{-t})}{(e^t + e^{-t})^2}$$

quotient rule

$$= \frac{e^{2t} + \overbrace{e^t e^{-t}}^{1} + \overbrace{e^{-t} e^t}^{1} + e^{-2t} - e^{2t} + \overbrace{e^t e^{-t}}^{1} + \overbrace{e^{-t} e^t}^{1} - e^{-2t}}{(e^t + e^{-t})^2}$$

$$= \frac{4}{(e^t + e^{-t})^2}$$

Since $e^{t}+e^{-t}>0$ we get the denominator is never zero or negative.

Since $(e^{t}+e^{-t})^2>0$ we get that

$$(\tanh(t))' = \frac{4}{(e^{t}+e^{-t})^2} > 0 \quad \text{for all } t\in\mathbb{R}.$$

Thus, $\tanh(t)$ is an increasing function. $\blacksquare$