Math 4300

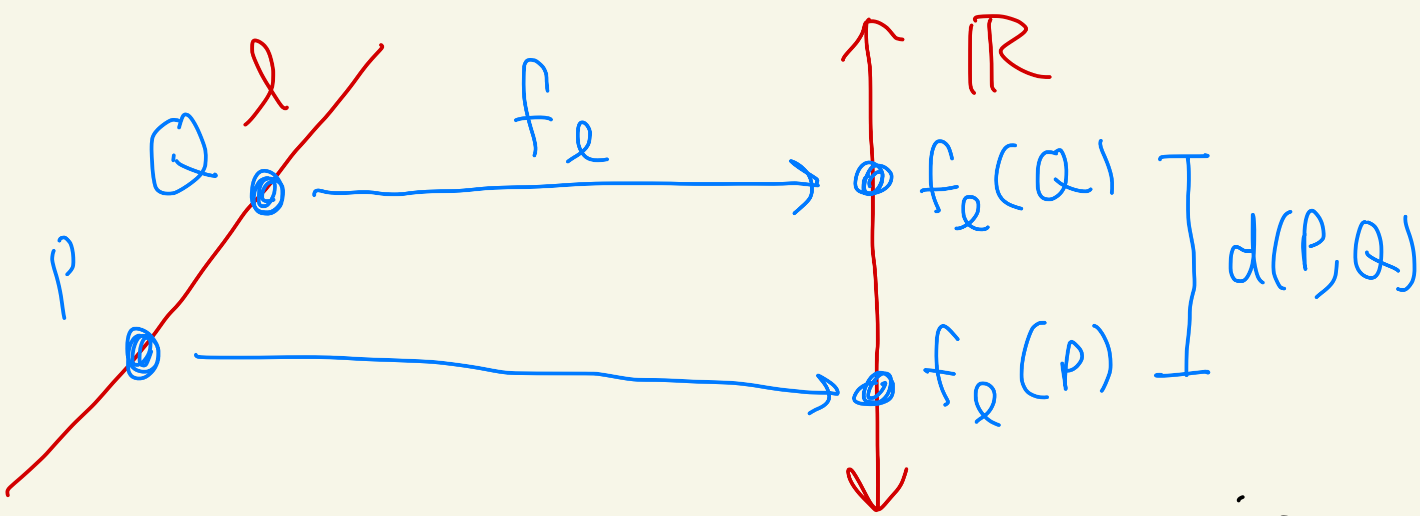
9/13/23

<u>Theorem:</u> Let $(\mathscr{P}, \mathscr{L})$ be an incidence geometry. Suppose for each line ℓ there exists a bijection $f_\ell : \ell \to \mathbb{R}$.

this will be a ruler for ℓ

Define $d: \mathscr{P} \times \mathscr{P} \to \mathbb{R}$ as follows: Given $P, Q \in \mathscr{P}$, let ℓ be the unique line ℓ through P and Q, and define

$$d(P,Q) = |f_\ell(P) - f_\ell(Q)|$$



The function d given above is a distance function and makes $(\mathscr{P}, \mathscr{L}, d)$ a metric geometry where f_ℓ is a ruler for ℓ.

<u>proof:</u> All we have to show is that d is a distance function. Let $P, Q \in \mathcal{P}$. Let ℓ be the unique line through P and Q

(i) $d(P,Q) = \underbrace{|f_\ell(P) - f_\ell(Q)|}_{\text{abs. value in } \mathbb{R}} \geq 0$

(ii) $d(P,Q) = 0$

iff $|f_\ell(P) - f_\ell(Q)| = 0$

iff $f_\ell(P) - f_\ell(Q) = 0$

iff $f_\ell(P) = f_\ell(Q)$

iff $P = Q$

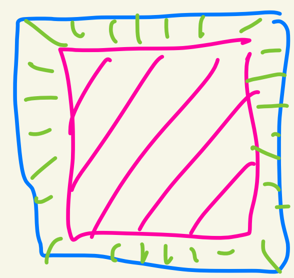
$\boxed{\begin{array}{c} x \in \mathbb{R} \\ |x| = 0 \\ \text{iff} \\ x = 0 \end{array}}$

$\boxed{\text{because } f_\ell \text{ is a bijection}}$

(iii) $d(P,Q) = |f_\ell(P) - f_\ell(Q)|$

$$= |f_\ell(Q) - f_\ell(P)|$$

$$= d(Q,P)$$



Let's apply this to the Hyperbolic plane $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$.

Let $P, Q \in \mathbb{H}$.

We want to define a distance function that respects the bijections that we made on Monday.

<u>Case 1:</u> Suppose $P=(a,y_1)$ and $Q=(a,y_2)$ both lie on ${}_aL$.

Recall that $g: {}_aL \to \mathbb{R}$ defined by $g(a,y)=\ln(y)$ was a bijection.

Define

$$d(P,Q) = |g(P) - g(Q)|$$

$$= |\ln(y_1) - \ln(y_2)|$$

$$= \left|\ln\left(\frac{y_1}{y_2}\right)\right|$$

Case 2: Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ both lie on ${}_cL_r$.

Recall that $f : {}_cL_r \to \mathbb{R}$ given by $f(x,y) = \ln\left(\frac{x-c+r}{y}\right)$ is a bijection

Define

$$d(P,Q) = |f(P) - f(Q)|$$

$$= \left|\ln\left(\frac{x_1-c+r}{y_1}\right) - \ln\left(\frac{x_2-c+r}{y_2}\right)\right|$$

$$= \left|\ln\left(\frac{\left(\frac{x_1-c+r}{y_1}\right)}{\left(\frac{x_2-c+r}{y_2}\right)}\right)\right|$$

Def: Given the hyperbolic plane $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$, define $d_H : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ as follows:

Let $P = (x_1, y_1)$, $Q = (x_2, y_2)$

Define

$$d_H(P,Q) = \begin{cases} \left| \ln\left(\dfrac{y_1}{y_2}\right) \right| & \text{if } P, Q \in {}_aL \text{ where } x_1 = x_2 = a \\[2ex] \left| \ln\left(\dfrac{\frac{x_1 - c + r}{y_1}}{\frac{x_2 - c + r}{y_2}} \right) \right| & \text{if } P, Q \in {}_cL_r \end{cases}$$

Theorem: $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H, d_H)$ is a metric geometry using the rulers

$f : {}_aL \to \mathbb{R}$ given by $f(a,y) = \ln(y)$

and

$f : {}_cL_r \to \mathbb{R}$ given by $f(x,y) = \ln\left(\frac{x+c-r}{y}\right)$

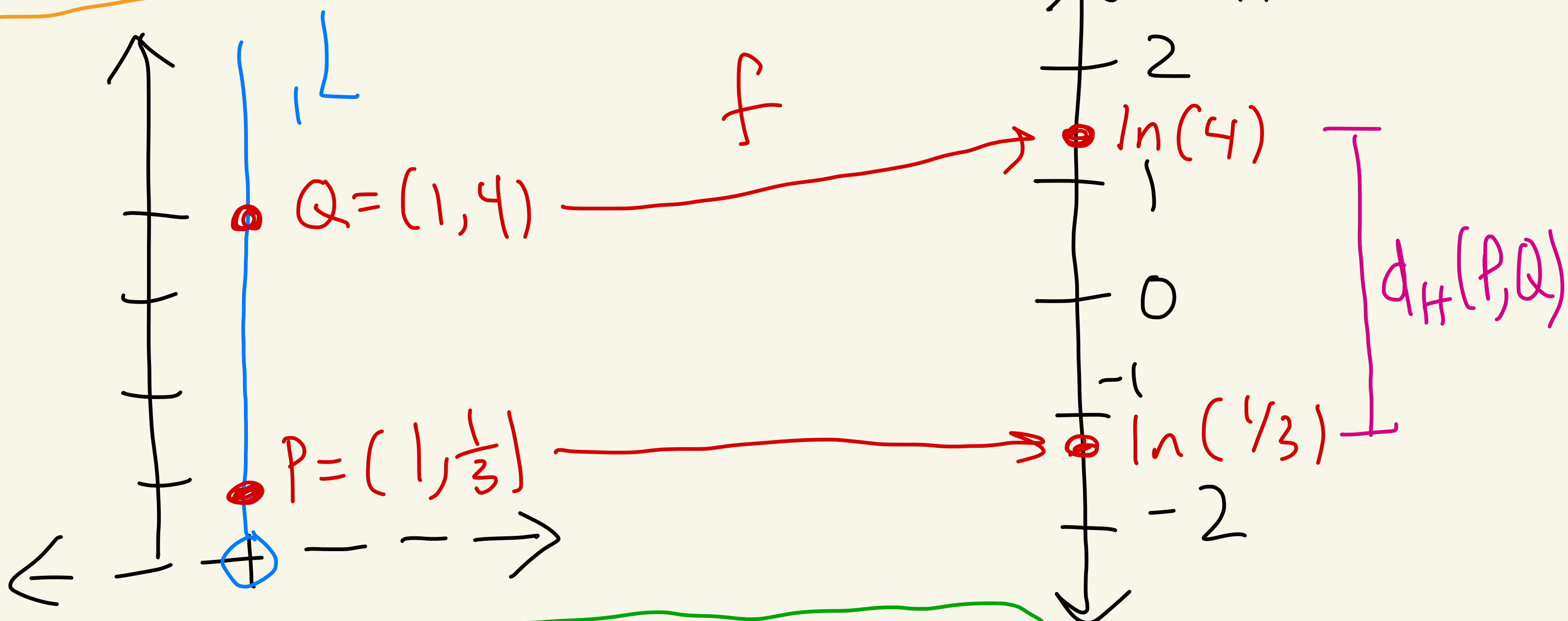
We call these the <u>standard rulers</u> for $\mathcal{H}$.

Proof: Already did it all. $\blacksquare$

Note: I changed g to f to simplify.

Ex: Let $P=(1,\frac{1}{3})$ and $Q=(1,4)$ be in the hyperbolic plane.

These points lie on ${}_1L$.



standard ruler:

$f(1,y)=\ln(y)$

$f(P)=\ln(\frac{1}{3})\approx -1.0986...$

$f(Q)=\ln(4)\approx 1.386...$

The coordinates of P, Q under the standard ruler are

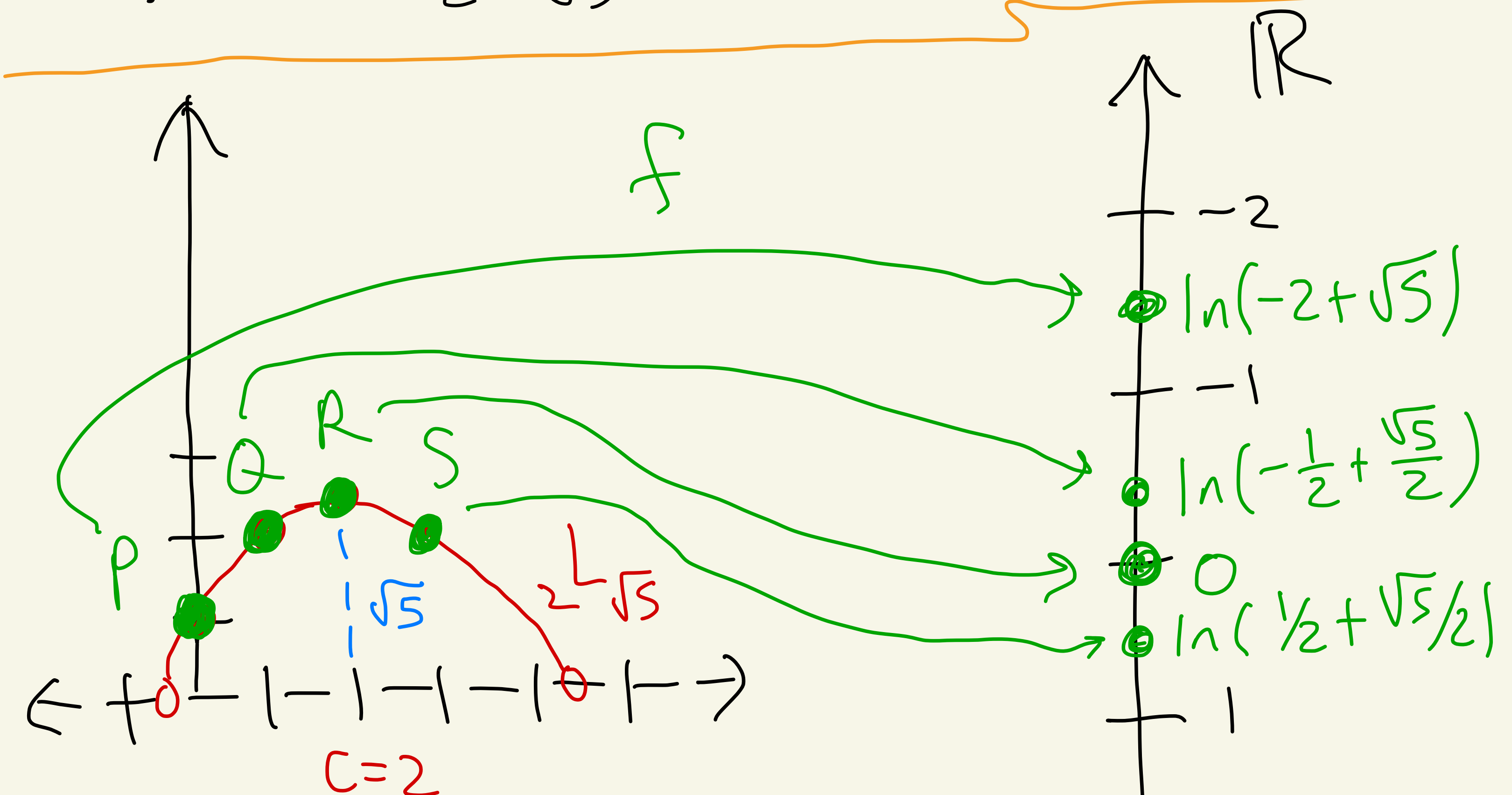
$f(P)=\ln(1/3)\approx -1.0986$

$f(Q)=\ln(4)\approx 1.386$

And

$$d_H(P,Q) = \underbrace{\left| \ln\left(\frac{1/3}{4}\right) \right| = \left| \ln\left(\frac{1}{12}\right) \right| \approx 2.4849}_{\text{same as } |\ln(1/3) - \ln(4)|}$$

Ex: Recall from Topic 1 that $P=(0,1)$, $Q=(1,2)$, $R=(2,\sqrt{5})$ all lie on ${}_2L_{\sqrt{5}}$. One can also show that $S=(3,2)$ is also on ${}_2L_{\sqrt{5}}$. $[\sqrt{5}\approx 2.23]$



$\boxed{\text{standard ruler}}$

$$f(x,y)=\ln\left(\frac{x-2+\sqrt{5}}{y}\right)$$

$$f(P)=\ln\left(\frac{0-2+\sqrt{5}}{1}\right)=\ln(-2+\sqrt{5})\approx -1.4436\ldots$$

$$f(Q) = \ln\left(\frac{1-2+\sqrt{5}}{2}\right) = \ln\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right) \approx -0.4812$$

$$f(R) = \ln\left(\frac{2-2+\sqrt{5}}{\sqrt{5}}\right) = \ln(1) = 0$$

$$f(S) = \ln\left(\frac{3-2+\sqrt{5}}{2}\right) = \ln\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right) \approx 0.4812$$

And

$$d_H(P,S) = |f(P) - f(S)| = \left|\ln(-2+\sqrt{5}) - \ln\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\right|$$

$$\approx 1.92485$$

same as

$$d_H(P,S) = \left| \ln \left(\frac{\frac{0-2+\sqrt{5}}{1}}{\frac{3-2+\sqrt{5}}{2}} \right) \right| \leftarrow$$

from d_H formula