Math 4300

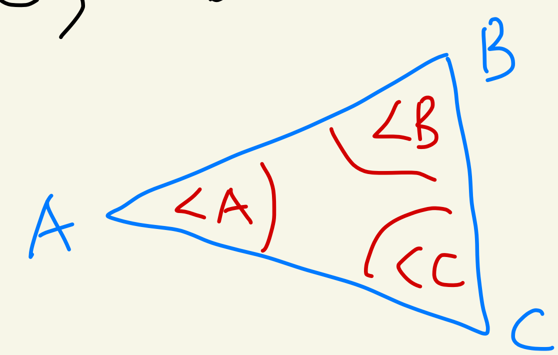
11/27/23

Topic 12 – Neutral Geometry

<u>Def:</u> Let $(\mathscr{P}, \mathscr{L}, d, m)$ be a protractor geometry.

- Two angles $\angle ABC$ and $\angle DEF$ are <u>congruent</u>, written $\angle ABC \cong \angle DEF$, if $m(\angle ABC) = m(\angle DEF)$

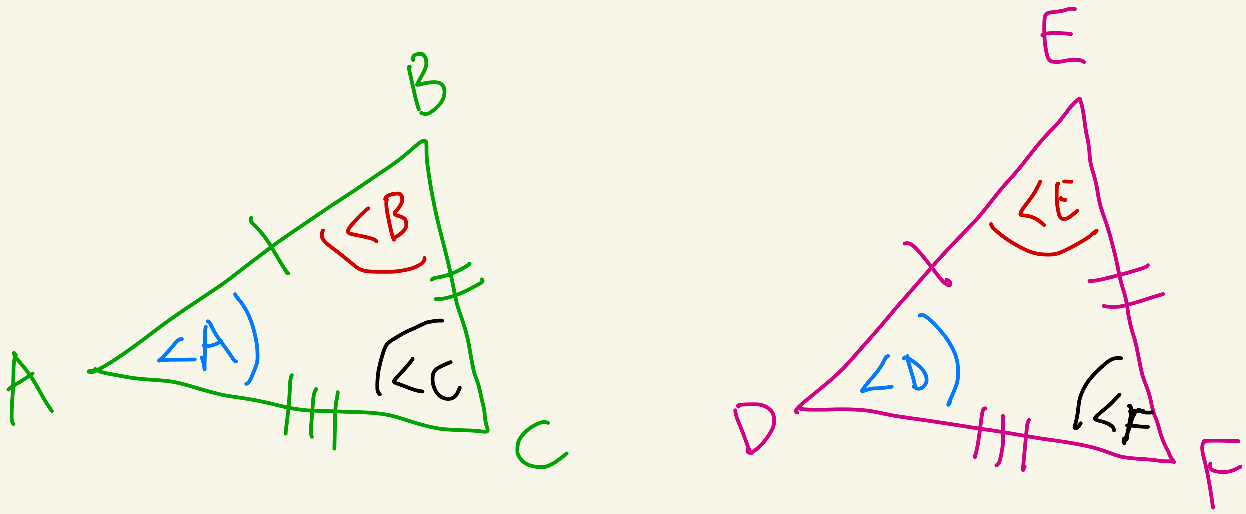
- Suppose we have a triangle $\triangle ABC$. We write $\angle A$ for $\angle BAC$, $\angle B$ for $\angle ABC$, and $\angle C$ for $\angle BCA$.



- Suppose $\triangle ABC$ and $\triangle DEF$ are triangles. We write $\triangle ABC \cong \triangle DEF$ if the following six conditions hold:

$\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$

$\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$



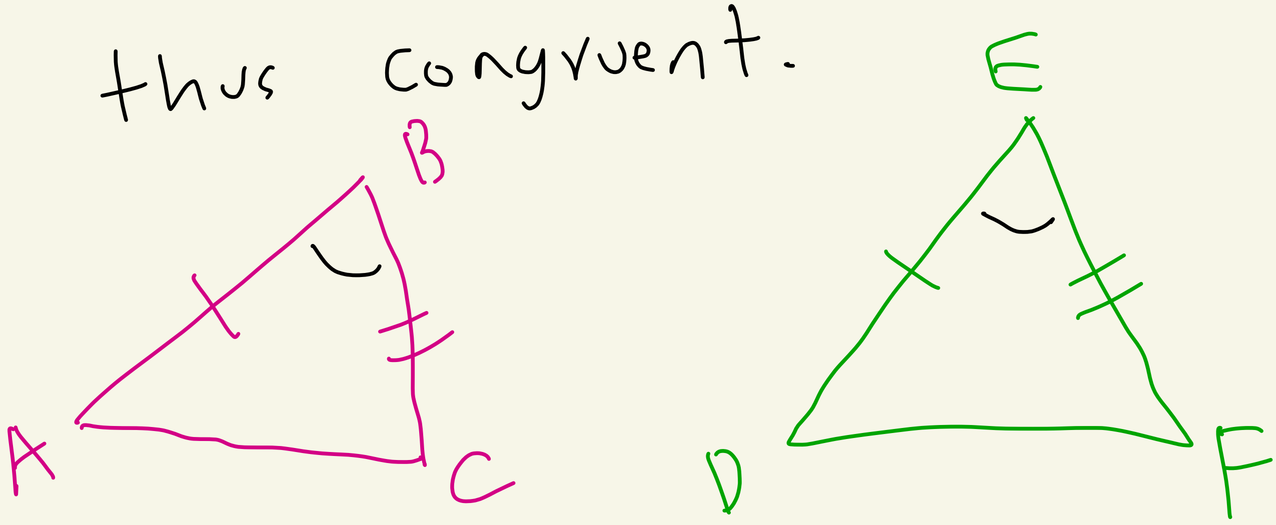
- Two triangles $\triangle ABC$ and $\triangle DEF$ are <u>congruent</u> if at least one of the following conditions hold:

$\triangle ABC \cong \triangle DEF$, $\triangle ACB \cong \triangle DEF$

$\triangle BCA \cong \triangle DEF$, $\triangle BAC \cong \triangle DEF$

$\triangle CAB \cong \triangle DEF$, $\triangle CBA \cong DEF$

<u>Def:</u> A protractor geometry $(\mathscr{P}, \mathscr{L}, d, m)$ satisfies the <u>side-angle-side axiom</u> (SAS) if whenever $\triangle ABC$ and $\triangle DEF$ are triangles with $\overline{AB} \simeq \overline{DE}$, $\angle B \simeq \angle E$, and $\overline{BC} \simeq \overline{EF}$, then $\triangle ABC \simeq \triangle DEF$ and are thus congruent.



A protractor geometry that satisfies SAS is called a <u>neutral geometry</u>.

Theorem: The Euclidean plane and the Hyperbolic plane are both neutral geometries.

proof: See Millman/Parker. $\blacksquare$

Ex: There is a protractor geometry called the taxicab plane that doesn't satisfy SAS. See Millman/Parker page 126-127.

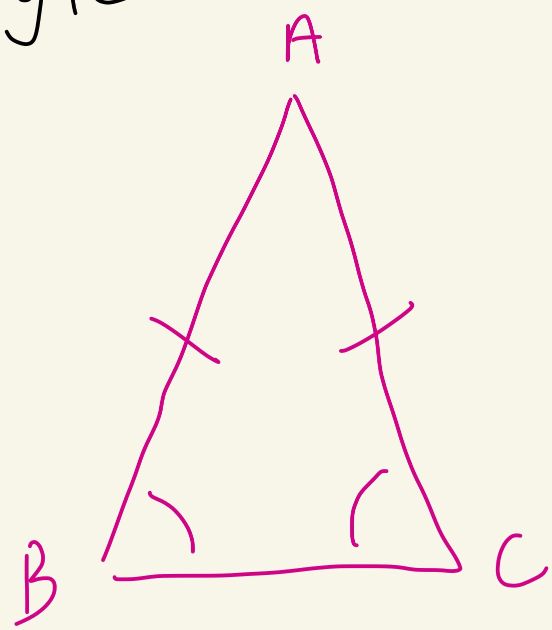
Def: Let $\triangle ABC$ be a triangle in a neutral geometry. $\triangle ABC$ is *isosceles* if at least two sides are congruent.

$\triangle ABC$ is called *equilateral* if all three sides are congruent.

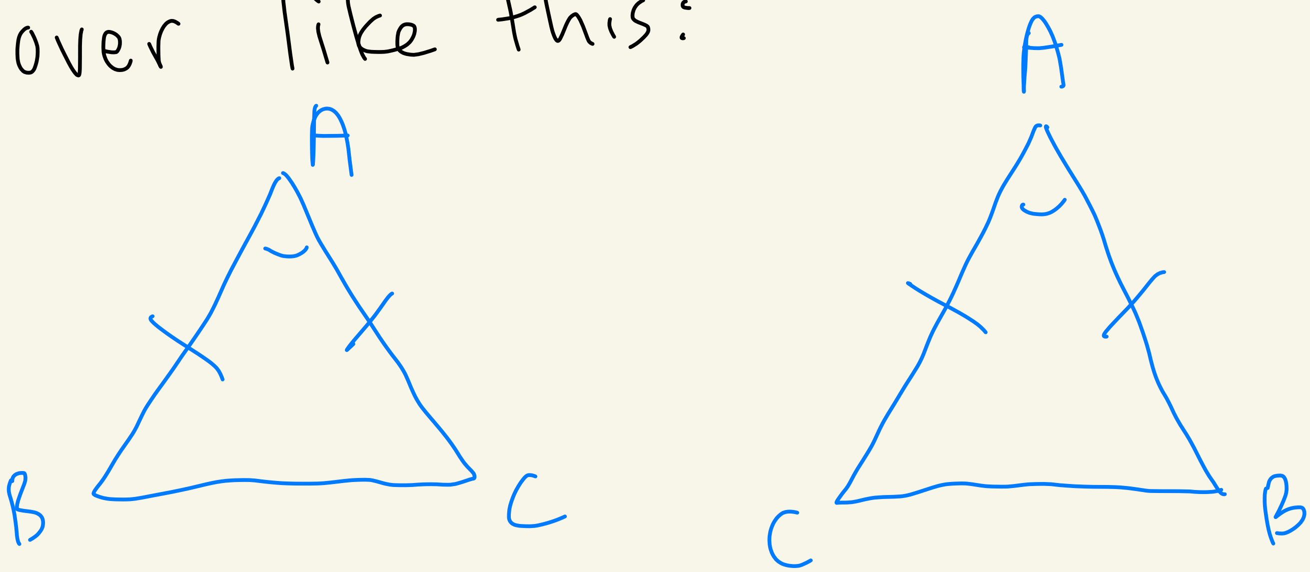
Theorem (Euclid) Let $\triangle ABC$ be an isosceles triangle in a neutral geometry.

Suppose $\overline{AB} \simeq \overline{AC}$.

Then, $\angle B \simeq \angle C$.



proof: Think of the same triangle twice but one flipped over like this:



We have that

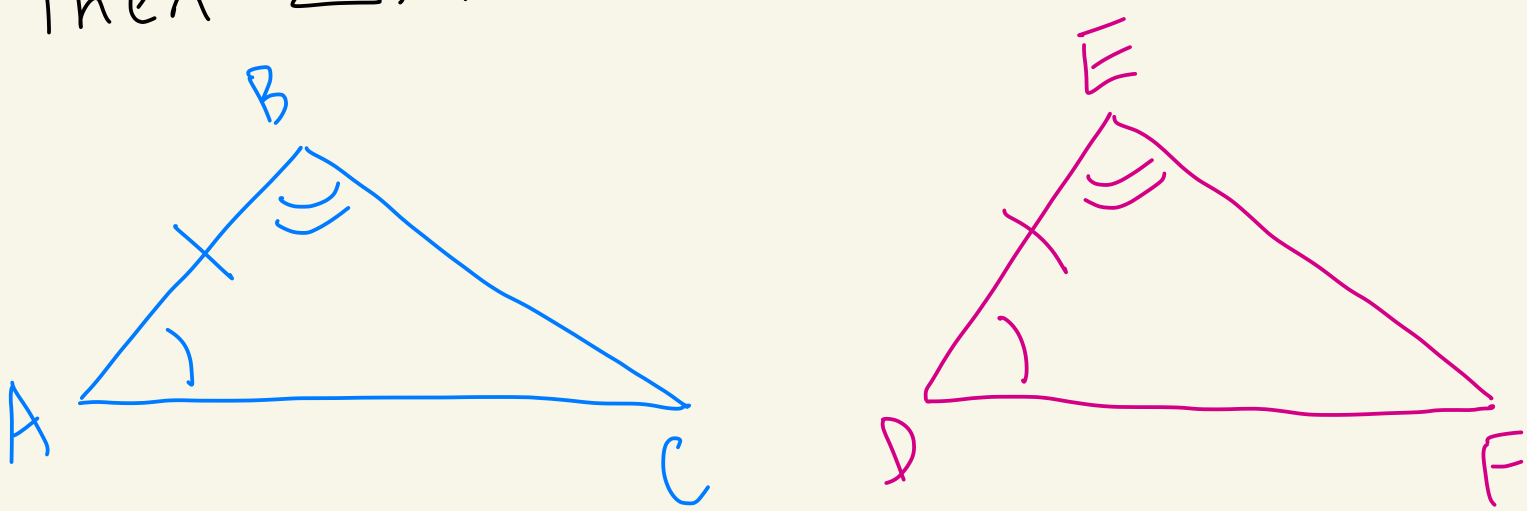
$\overline{AB} \cong \overline{AC}$, $\angle A \cong \angle A$, $\overline{AC} \cong \overline{AB}$.

By SAS we get $\triangle BAC \cong \triangle CAB$.

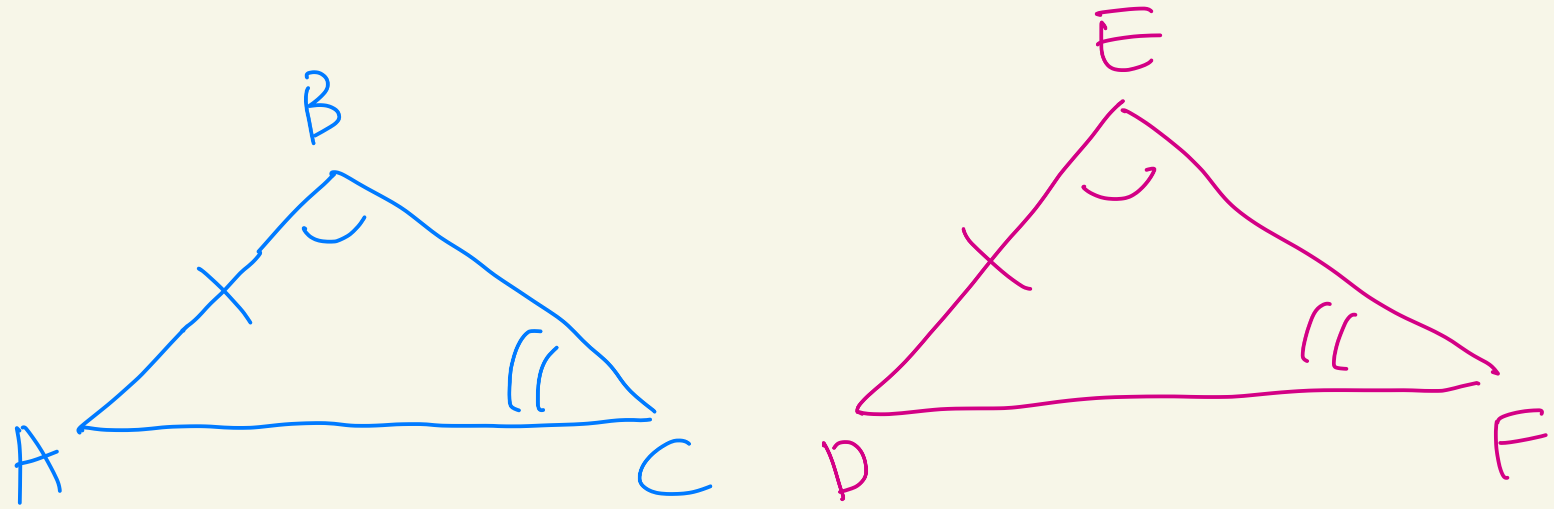
Then, $\angle B \cong \angle C$. $\blacksquare$

Def: Let $G = (\mathscr{P}, \mathscr{L}, d, m)$ be a protractor geometry.

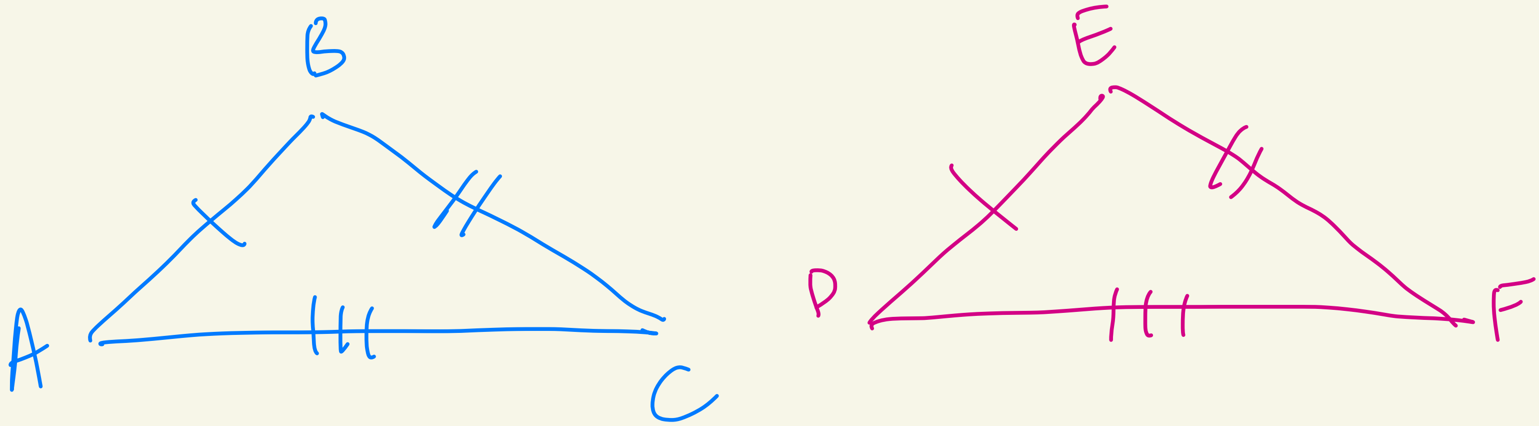
(i) G satisfies angle-side-angle (ASA) if given any two triangles $\triangle ABC$ and $\triangle DEF$ with $\angle A \cong \angle D$, $\overline{AB} \cong \overline{DE}$, and $\angle B \cong \angle E$, then $\triangle ABC \cong \triangle DEF$



(ii) G satisfies side-angle-angle (SAA) if given any two triangles $\triangle ABC$ and $\triangle DEF$ with $\overline{AB} \cong \overline{DE}$, $\angle B \cong \angle E$, and $\angle C \cong F$, then $\triangle ABC \cong \triangle DEF$



(iii) G satisfies side-side-side (SSS) if given any two triangles $\triangle ABC$ and $\triangle DEF$ with $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$, then $\triangle ABC \cong \triangle DEF$.

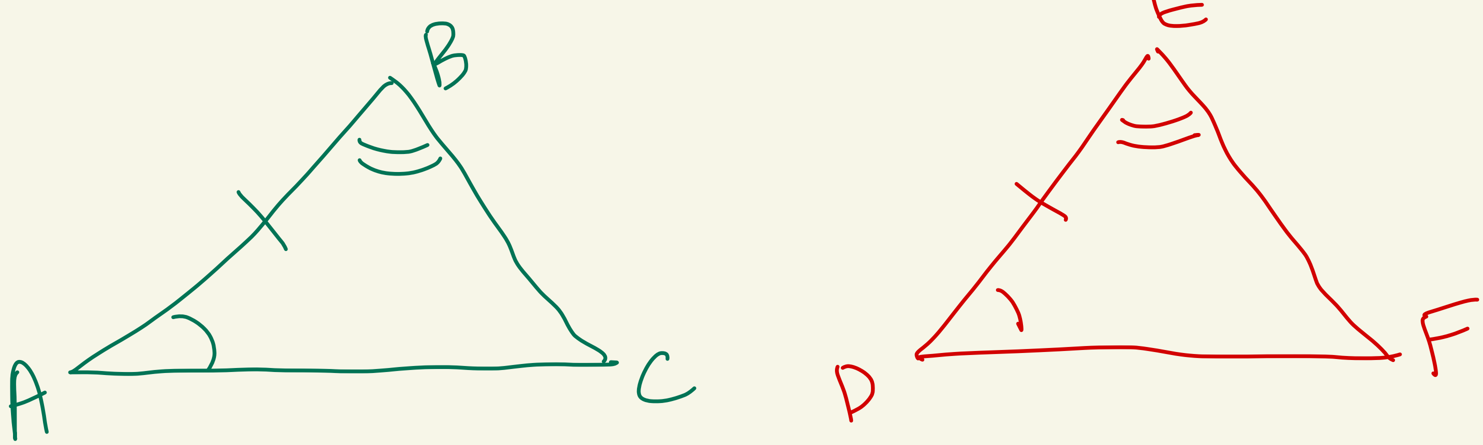


END OF DEF

<u>Theorem:</u> Let $G = (\mathcal{P}, \mathcal{L}, d, m)$ be a neutral geometry.
Then G satisfies ASA, SAA, and SSS.

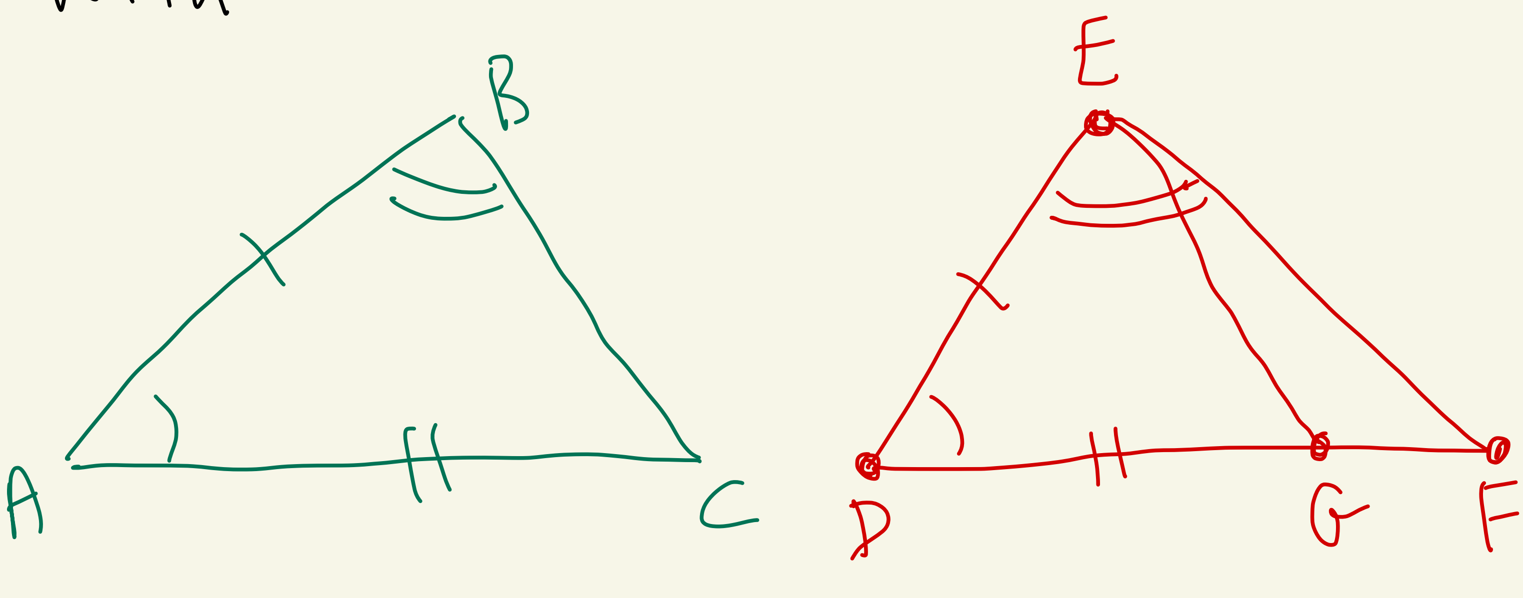
<u>Proof:</u> We will prove ASA.
See Millman / Parker for other two.
Suppose we have two triangles

$\triangle ABC$ and $\triangle DEF$ with
$\angle A \cong \angle D$, $\overline{AB} \cong \overline{DE}$, and $\angle B \cong \angle E$



We will show that $\triangle ABC \cong \triangle DEF$.

By the segment construction theorem there exists a unique point $G \in \overrightarrow{DF}$ with $\overline{DG} \cong \overline{AC}$.



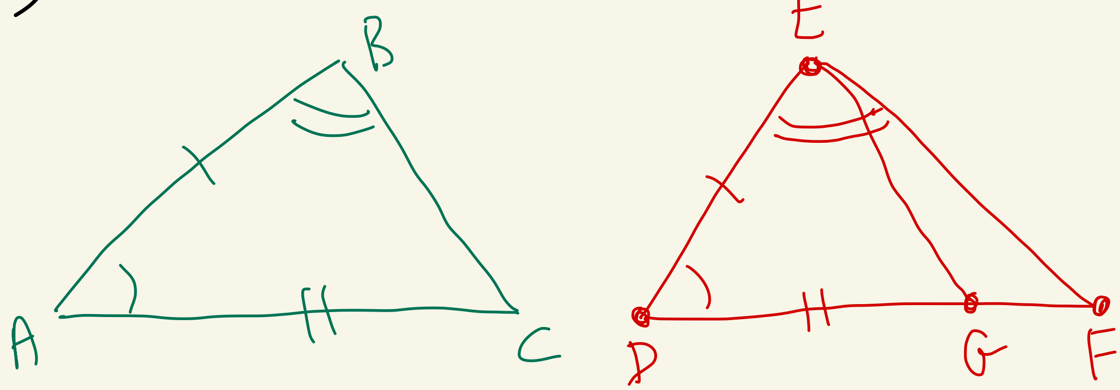
We will show that $\triangle ABC \cong \triangle DEG$ and then $G = F$ to get $\triangle ABC \cong \triangle DEF$.

Since $\overline{AB} \cong \overline{DE}$, $\angle A \cong \angle D$, and $\overline{AC} \cong \overline{DG}$, by SAS we get $\triangle ABC \cong \triangle DEG$.

Thus, $\angle ABC \cong \angle DEG$.

But we know also $\angle ABC \cong \angle DEF$.

Thus, $\angle DEG \cong \angle DEF$.



Since $G \in \overrightarrow{DF}$ we know G and F lie in the same half-plane determined by $\overleftrightarrow{DE}$.

[Hw 9 #1(b)]

By (iii) of angle measure there is a unique ray $\overrightarrow{EF} = \overrightarrow{EG}$ with $m(\angle DEG) = m(\angle DEF)$.

Thus,

$$\{F\} = \overrightarrow{EF} \cap \overrightarrow{DF} = \overrightarrow{EG} \cap \overrightarrow{DF} = \{G\}$$

So, $F = G$.

Thus, $\triangle BAC \cong \triangle EDG \cong \triangle EDF$.

So, $\triangle ABC \cong \triangle DEF$. $\blacksquare$