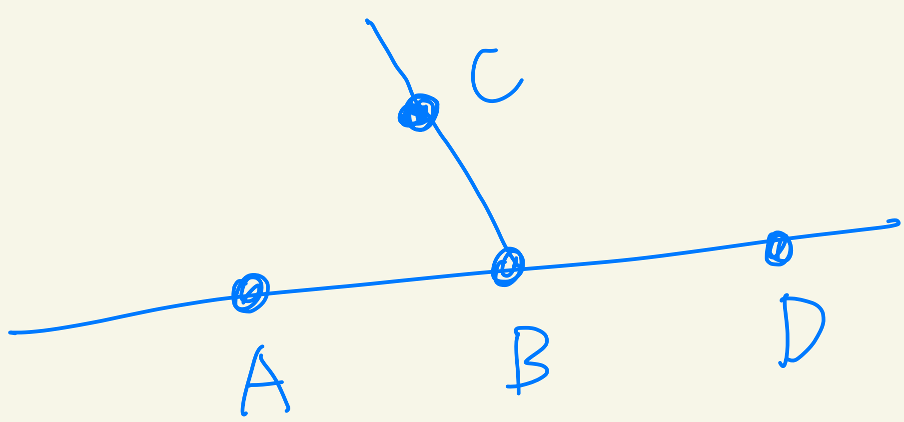
Math 4300
11/13/23

Topic 11 - More on angles

<u>Def:</u> Let $(\mathscr{P}, \mathscr{L}, d, m)$ be a protractor geometry.

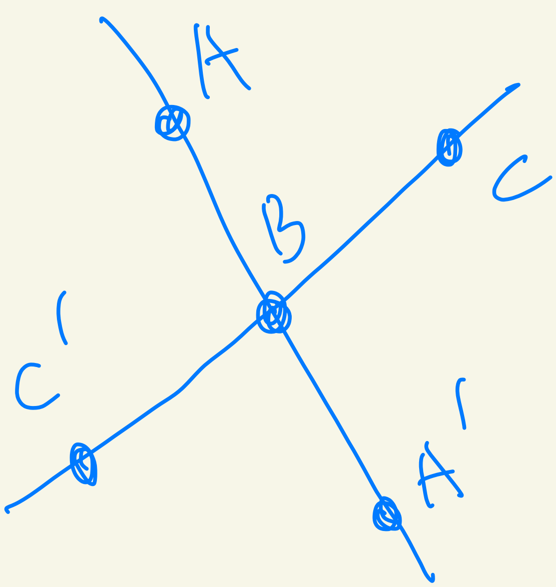
- Two angles are <u>supplementary</u> if their angle sum is $180°$.
- Two angles are <u>complementary</u> if their angle sum is $90°$
- Two angles $\angle ABC$ and $\angle CBD$ form a <u>linear pair</u> if $A-B-D$.



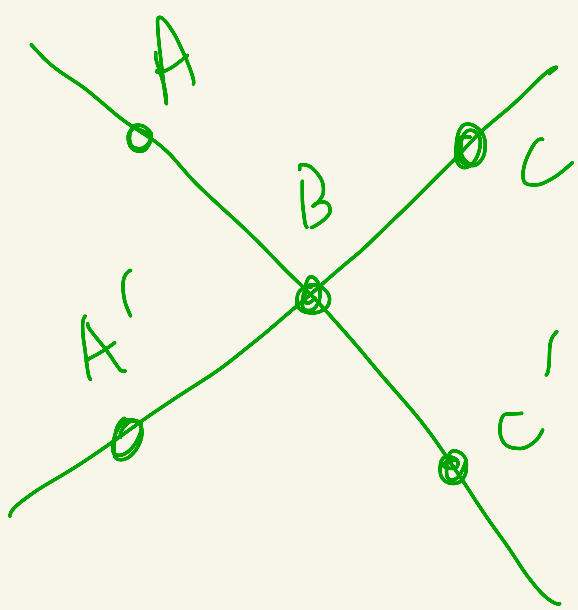
- Two angles $\angle ABC$ and $\angle A'BC'$ form a <u>vertical pair</u> if their union is a pair of intersecting lines. Another way to say this is if either

(i) $A - B - A'$ and $C - B - C'$

OR (ii) $A - B - C'$ and $C - B - A'$

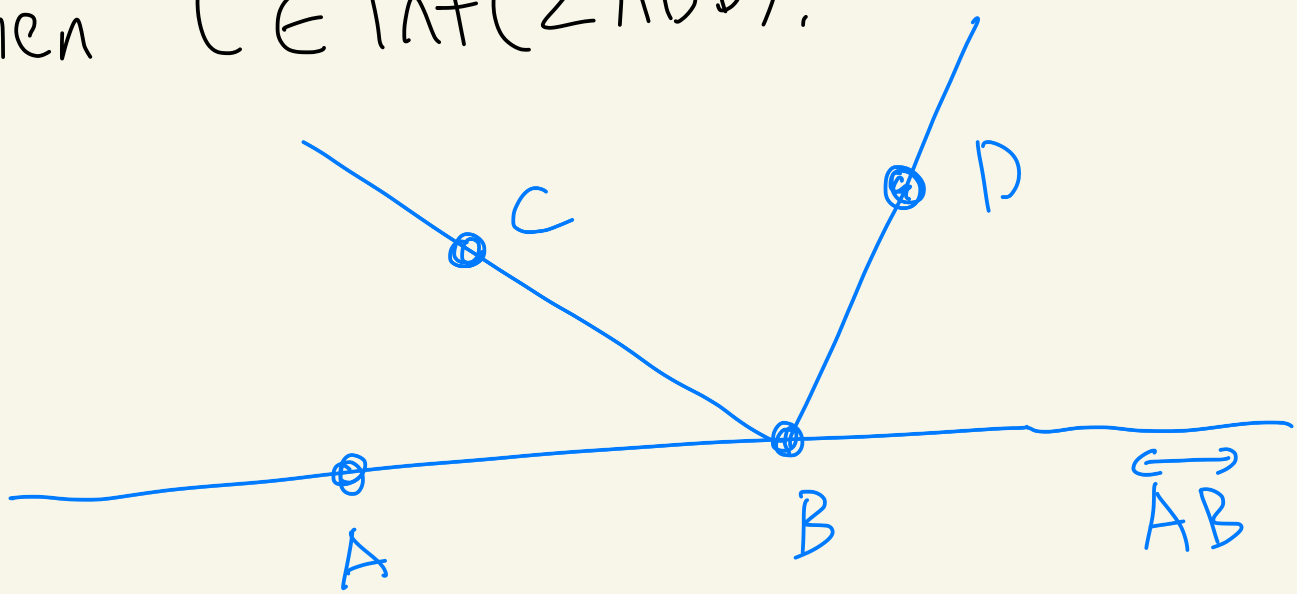


case (i)



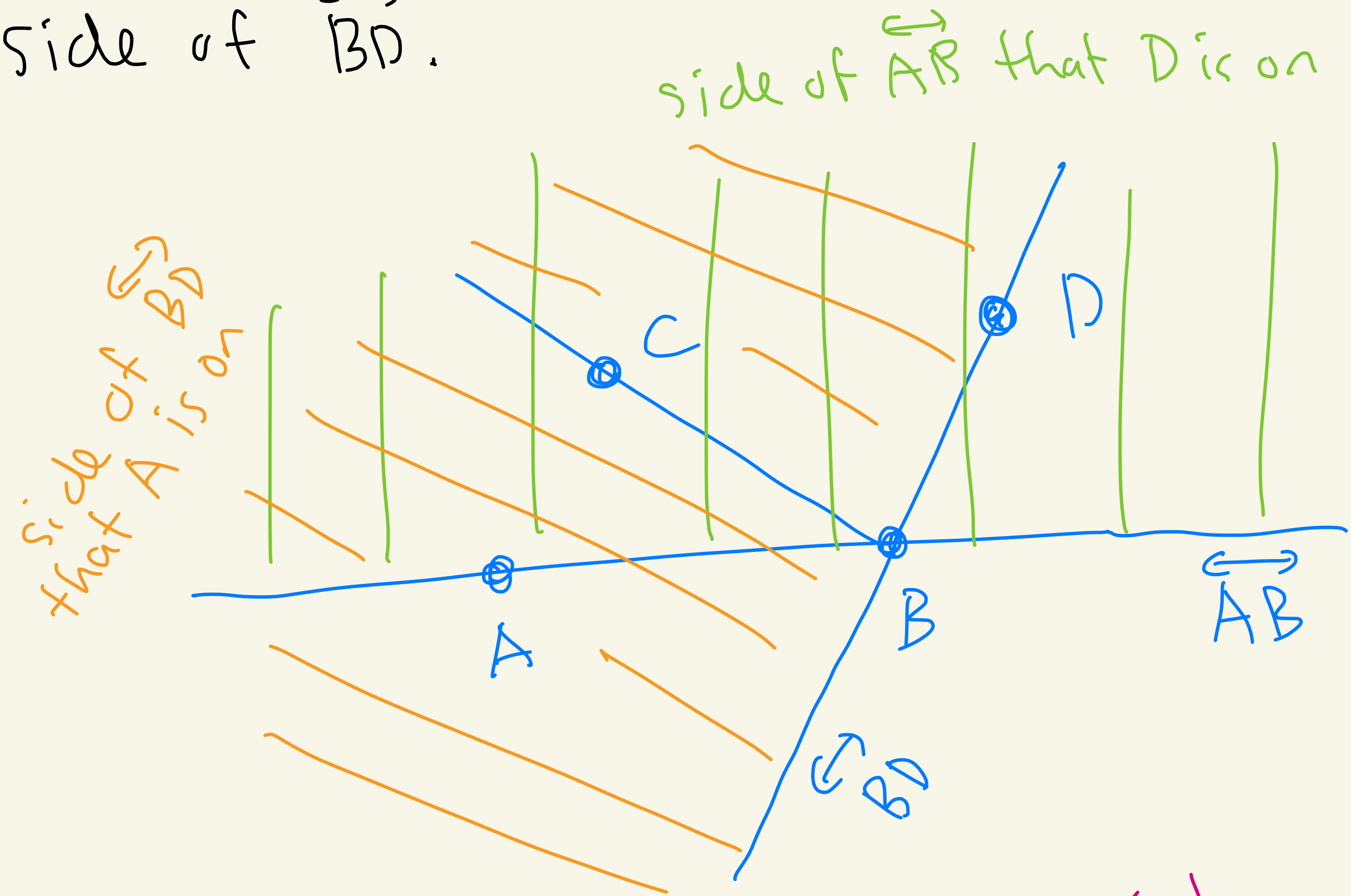
case (ii)

<u>Lemma 1:</u> Let $(\mathscr{P}, \mathscr{L}, d, m)$ be a protractor geometry. Let $\angle ABC$ and $\angle ABD$ be angles. If C and D lie on the same side of $\overleftrightarrow{AB}$ and $m(\angle ABC) < m(\angle ABD)$, then $C \in \operatorname{int}(\angle ABD)$.



<u>proof:</u> Since C and D lie on the same side of $\overleftrightarrow{AB}$, to show that $C \in \operatorname{int}(\angle ABD)$, we just need to show that C and A are on the same

side of $\overleftrightarrow{BD}$.



(intersection of above half-planes is int($\angle ABD$))

To do this we rule out the other two cases: $C \in \overleftrightarrow{BD}$, or C and A are on opposite sides of $\overleftrightarrow{BD}$.

<u>Case 1</u>: Suppose $C \in \overleftrightarrow{BD}$.

Claim: This implies that $\overrightarrow{BD} = \overrightarrow{BC}$.

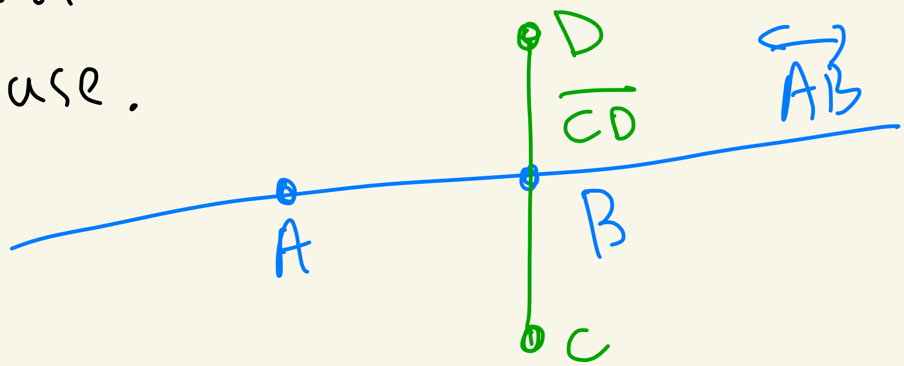
Why?

We know C and D lie on the same side of $\overleftrightarrow{AB}$.

We know $C \neq B$.

Since $C \in \overleftrightarrow{BD}$ either $C-B-D$, or $B-C-D$ or $C=D$ or $B-D-C$.

If $C-B-D$ then $\overline{CD} \cap \overleftrightarrow{AB} \neq \emptyset$ since $B \in \overline{CD} \cap \overleftrightarrow{AB}$. Then C and D are on opposite sides of $\overleftrightarrow{AB}$ which isn't the case.



The other 3 cases gives $C \in \overrightarrow{BD}$ and by HW 5 problem 9 we get $\overrightarrow{BC} = \overrightarrow{BD}$

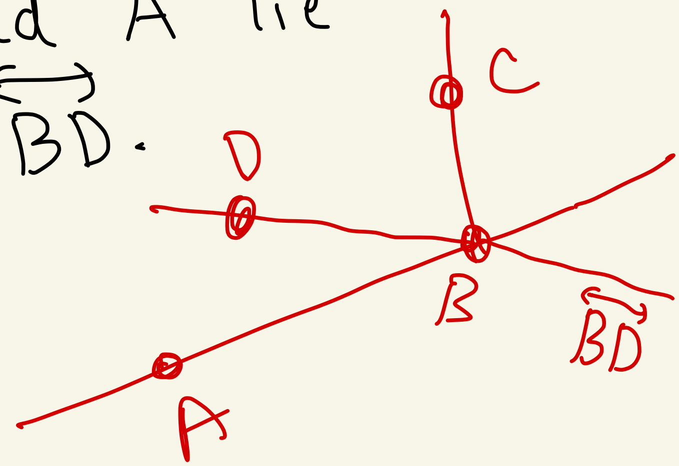
$\boxed{\text{Claim}}$

This gives $\angle ABC = \angle ABD$ and

$$m(\angle ABC) = m(\angle ABD)$$

which is a contradiction.

<u>Case 2:</u> Suppose C and A lie on opposite sides of $\overleftrightarrow{BD}$.



Since by assumption, C and D lie on the same side of $\overleftrightarrow{AB}$, by Hw 9 #2, we get that $D \in \operatorname{int}(\angle ABC)$

Then

property (iii) of m

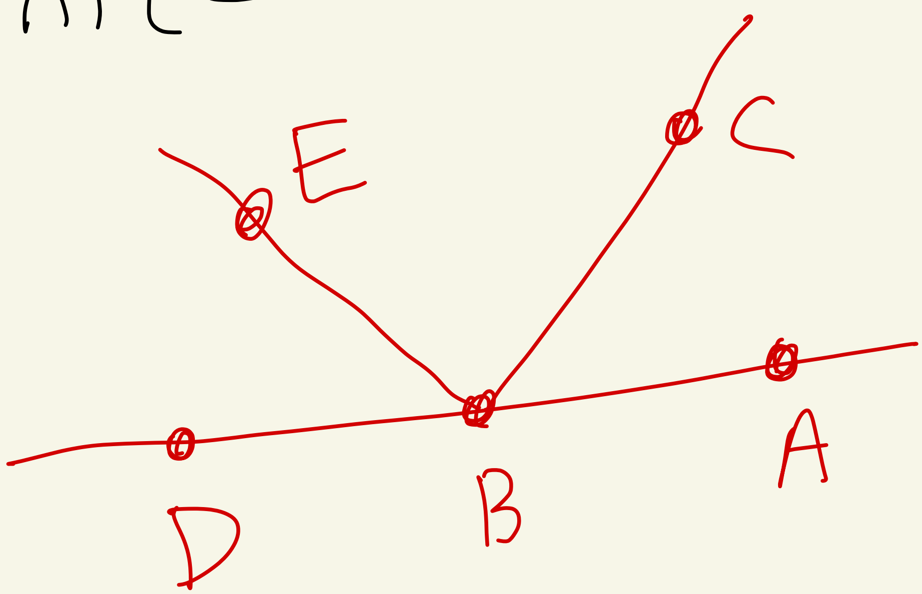
$$m(\angle ABD) + m(\angle DBC) = m(\angle ABC) < m(\angle ABD).$$

Then $m(\angle DBC) < 0$.

Contradiction.

∎

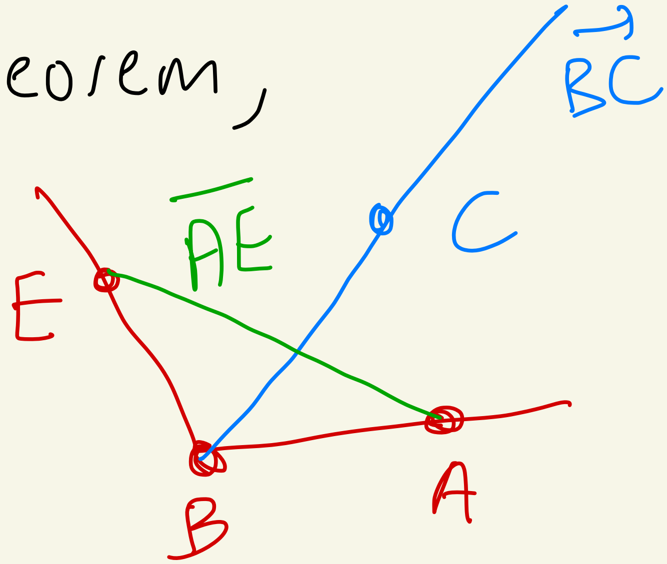
<u>Lemma 2</u>: Let $(\mathcal{P}, \mathcal{L}, d, m)$ be a protractor geometry. If $A-B-D$ and $C \in \operatorname{int}(\angle ABE)$ then $E \in \operatorname{int}(\angle CBD)$



<u>proof</u>: Suppose $A-B-D$ and $C \in \operatorname{int}(\angle ABE)$.

Since $C \in \operatorname{int}(\angle ABE)$ we know that C and E are on the same side of $\overleftrightarrow{AB} = \overleftrightarrow{BD}$ $(*)$

By the crossbar theorem,
$\overrightarrow{BC} \cap \overline{AE} \neq \phi$.



Thus A and E are on opposite sides of $\overleftrightarrow{BC}$. (1)

Since A-B-D we know $\underbrace{\overline{AD} \cap \overleftrightarrow{BC}}_{\text{B is in here}} \neq \phi$

So A and D are on opposite sides of $\overleftrightarrow{BC}$. (2)

By (1) and (2) we get E and D are on the same side of $\overleftrightarrow{BC}$. (HW 7 #7) (**)

(*) and (**) give $E \in \operatorname{int}(\angle CBD)$

$\blacksquare$