Math 4300

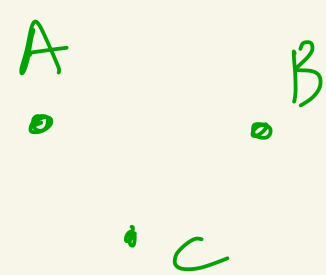
10/9/23

HW 1

(9) Let $(\mathcal{P}, \mathcal{L})$ be an incidence geometry. Let P be a point. Prove there exists a line ℓ where $P \notin \ell$.

<u>proof:</u> Suppose otherwise. That is, suppose P lies on every line in $\mathcal{L}$.

Since $(\mathcal{P}, \mathcal{L})$ is an incidence geometry there exists distinct points A, B, C that are non-collinear.



<u>case 1:</u> Suppose $P \in \{A, B, C\}$.

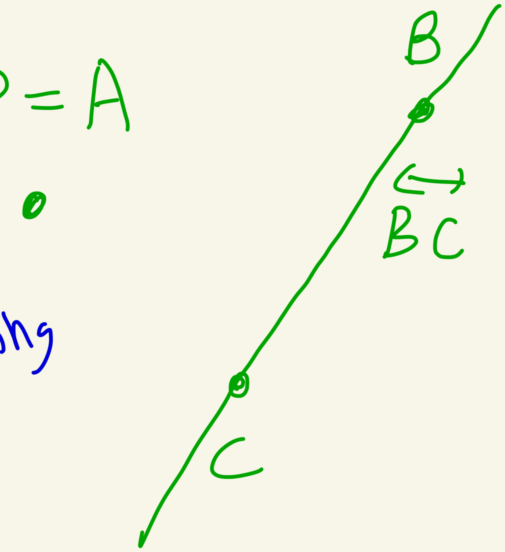
Without loss of generality, assume $P = A$.

By assumption, $P \in \overleftrightarrow{BC}$

But then $A = P \in \overleftrightarrow{BC}$.

Then, $A, B, C \in \overleftrightarrow{BC}$ contradicting that they are collinear.

$P = A$



Case 2: Suppose $P \notin \{A, B, C\}$.

Claim: Either $P \notin \overleftrightarrow{AB}$ or $P \notin \overleftrightarrow{AC}$, or $P \notin \overleftrightarrow{BC}$.

pf of claim: We just have to rule out the case where $P \in \overleftrightarrow{AB}$, $P \in \overleftrightarrow{AC}$, and $P \in \overleftrightarrow{BC}$.

Suppose $P \in \overleftrightarrow{AB}$ and $P \in \overleftrightarrow{AC}$ and $P \in \overleftrightarrow{BC}$.

Note that $A, P \in \overleftrightarrow{AB}$.

Also, $A, P \in \overleftrightarrow{AC}$.

But there is a unique line through A and P.

Thus, $\overleftrightarrow{AB} = \overleftrightarrow{AC}$.

But then $A, B, C \in \overleftrightarrow{AB}$

since $\overleftrightarrow{AB} = \overleftrightarrow{AC}$

which contradicts that A, B, C are non-collinear $\boxed{\text{claim}}$

By case 1 and case 2, there has to be a line that P is not on.

HW 1

(10) Let $(\mathcal{P}, \mathcal{L})$ be an incidence geometry. Let P be any point. Prove there exists points Q, R where P, Q, R are non-collinear.

<u>Proof by contradiction:</u>

Suppose given any two points Q, R we have that P, Q, R are collinear.

Since we have an incidence geometry there must exist distinct points A, B, C that are non-collinear.

Case 1: Suppose $P = A$.

Then, P, B, C are non-collinear which contradicts our assumption.

Case 2: Suppose $P \neq A$.

By assumption, P, A, B are collinear and hence $P, A, B \in \overleftrightarrow{AB}$.

By assumption, P, A, C are collinear and hence $P, A, C \in \overleftrightarrow{AC}$.

Then, $P, A \in \overleftrightarrow{AB}$ and $P, A \in \overleftrightarrow{AC}$.

Since there is a unique line through any two distinct points we know $\overleftrightarrow{AB} = \overleftrightarrow{AC}$

But then $A, B, C \in \overleftrightarrow{AB}$ which contradicts that A, B, C are non-collinear.

Both cases lead to contradictions, so we are done. $\blacksquare$

$\boxed{\text{HW 2}}$ Hyperbolic plane

7 $P=(2,3)$, $Q=(-1,6)$

Find ruler f on $\overleftrightarrow{PQ}$ where $f(P)=0$ and $f(Q)>0$

Plug P, Q into $(x-c)^2+y^2=r^2$.

$$\boxed{\begin{aligned}(2-c)^2+3^2&=r^2 \quad \leftarrow P\\ (-1-c)^2+6^2&=r^2 \quad \leftarrow Q\end{aligned}}$$

$\Downarrow$

$$\boxed{\begin{aligned}c^2-4c+13&=r^2 \quad (1)\\ c^2+2c+37&=r^2 \quad (2)\end{aligned}}$$

$\Downarrow$ (1) − (2)

$$\boxed{\begin{aligned}-6c-24&=0\\ c&=-4\end{aligned}} \Rightarrow$$

plug into (1) to get

$$\begin{aligned}r^2&=(-4)^2-4(-4)+13\\ &=16+16+13\\ &=45\\ r&=\sqrt{45}=3\sqrt{5}\end{aligned}$$

Thus, $P, Q \in \underbrace{{}_{-4}L_{3\sqrt{5}}}_{{}_{c}L_{r}}$

The standard ruler for ${}_{-4}L_{3\sqrt{5}}$ is

$$f(x,y) = \ln\left(\frac{x-c+r}{y}\right) = \ln\left(\frac{x+4+3\sqrt{5}}{y}\right)$$

$$f(P) = f(2,3) = \ln\left(\frac{2+4+3\sqrt{5}}{3}\right)$$
$$= \ln\left(\frac{6+3\sqrt{5}}{3}\right) \approx 1.44$$

$$f(Q) = f(-1,6) = \ln\left(\frac{-1+4+3\sqrt{5}}{6}\right)$$
$$= \ln\left(\frac{3+3\sqrt{5}}{6}\right) \approx 0.481$$

Since $f(Q) < f(P)$ we need to shift
<u>and</u> the flip!

Ruler we want

$$g(x,y) = \underset{\substack{\uparrow \\ \text{flip}}}{-}\Big(\underbrace{f(x,y) - f(P)}_{\text{shift}}\Big)$$

$$\boxed{\begin{aligned} &= -\left(\ln\left(\frac{x+4+3\sqrt{5}}{y}\right) - \ln\left(\frac{6+3\sqrt{5}}{3}\right)\right) \\ &= -\ln\left(\frac{\frac{x+4+3\sqrt{5}}{y}}{\frac{6+3\sqrt{5}}{3}}\right) \\ &= \ln\left(\frac{\frac{6+3\sqrt{5}}{3}}{\frac{x+4+3\sqrt{5}}{y}}\right) \end{aligned}}$$