Math 4300

10/2/23

# Plan

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|---|---|
| 10/2 – Topic 5 | 10/4 – Topic 5/6 |
| 10/9 HW Review | 10/11 Test 1 |

On test put

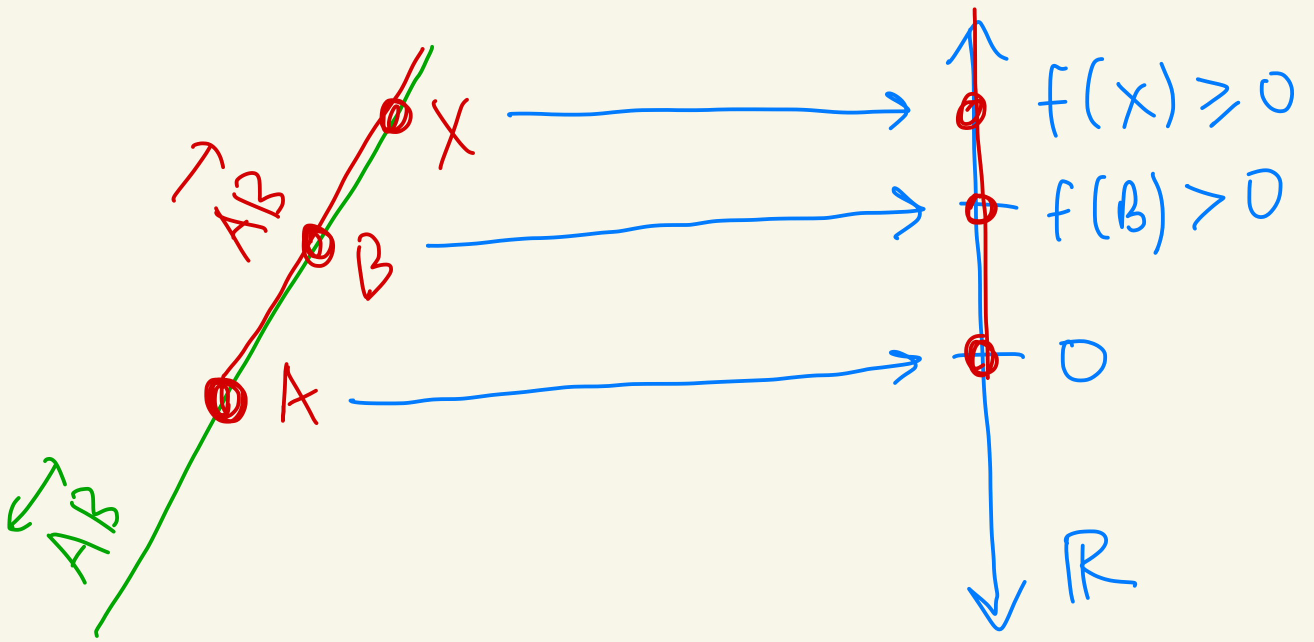
$$f(x,y) = \ln\left(\frac{x-c+r}{y}\right)$$

$\boxed{\text{Theorem:}}$ Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry. Let $A, B \in \mathscr{P}$ with $A \neq B$.

Then there exists a ruler $f : \overleftrightarrow{AB} \to \mathbb{R}$ where

$$\overrightarrow{AB} = \left\{ X \in \overleftrightarrow{AB} \mid f(X) \geq 0 \right\}$$

One such ruler is one where $f(A) = 0$ and $f(B) > 0$



proof: By a previous theorem there exists a ruler $f: \overleftrightarrow{AB} \to \mathbb{R}$ where $f(A) = 0$ and $f(B) > 0$.

Claim: $\overrightarrow{AB} = \{ X \in \overleftrightarrow{AB} \mid f(X) \geq 0 \}$

$\boxed{\subseteq}$: Let $X \in \overrightarrow{AB}$.

Then, $X \in \overleftrightarrow{AB}$ because $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$

We need to show that $f(X) \geq 0$.

Suppose instead that $f(X) < 0$.

Then, $f(X) < \underbrace{f(A)}_{0} < f(B)$.

Thus, $X - A - B$.

But $\overrightarrow{AB} = \left\{ C \in \mathscr{P} \;\middle|\; \begin{array}{l} C = A \text{ or } A-C-B \\ \text{or } C = B \text{ or } A-B-C \end{array} \right\}$

and Hw #4 says one and only one of the following can be true for X:

$$\underbrace{X - A - B}_{X \notin \overrightarrow{AB}} \text{ or } \underbrace{X = A \text{ or } A - X - B \text{ or } X = B \text{ or } A - B - X}_{X \in \overrightarrow{AB}}$$

So, $X - A - B$ implies $X \notin \overrightarrow{AB}$ which is a contradiction.

Thus, $f(X) \geq 0$

So, $X \in \{ C \in \mathcal{P} \mid f(C) \geq 0 \}$.

$\boxed{\supseteq}$: Let $X \in \{ C \in \mathcal{P} \mid f(C) \geq 0 \}$

We need to show $X \in \overrightarrow{AB}$.

We have $f(X) \geq 0$.

If $f(X) = 0$, then since $f$ is 1-1 and $f(A) = 0$ we have that

$X = A$ which implies $X \in \overrightarrow{AB}$.

If $\underbrace{0}_{f(A)} < f(X) < f(B)$, then

$A - X - B$ and so $X \in \overrightarrow{AB}$.

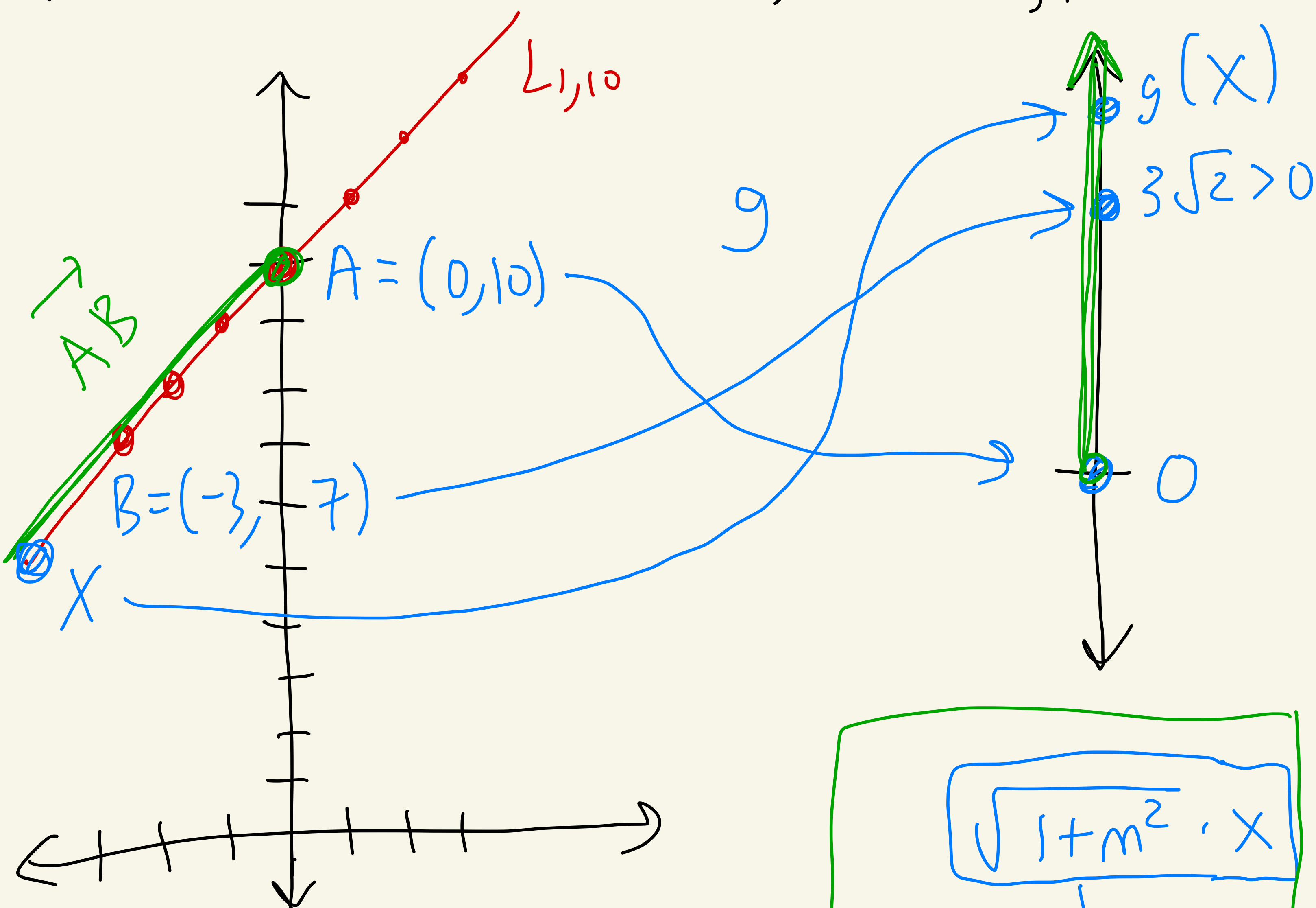
If $f(X) = f(B)$, then since $f$ is 1-1 we have that $X = B$ and so $X \in \overrightarrow{AB}$.

If $\underbrace{0}_{f(A)} < f(B) < f(X)$, then

$A - B - X$ and so $X \in \overrightarrow{AB}$.

That's all the cases, so $X \in \overrightarrow{AB}$.

<u>Ex:</u> Consider $L_{m,b} = L_{1,10}$




$\boxed{\sqrt{1+m^2}\cdot X}$

<u>Standard ruler:</u> $f(x,y) = \sqrt{2}\,x$

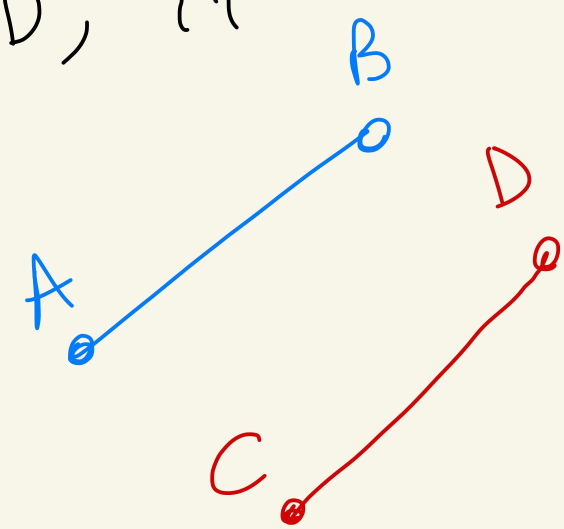
$f(A) = 0$, $f(B) = -3\sqrt{2} < 0$

<u>set:</u> $g(X) = -f(X)$

Then, $g(A) = 0$, $g(B) = 3\sqrt{2} > 0$

<u>Def:</u> Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry.
Let $A, B, C, D \in \mathscr{P}$ with $A \neq B$ and $C \neq D$.
We say that the line segments $\overline{AB}$ and $\overline{CD}$ are congruent, and write $\overline{AB} \cong \overline{CD}$, if

$$\underbrace{AB = CD}.$$



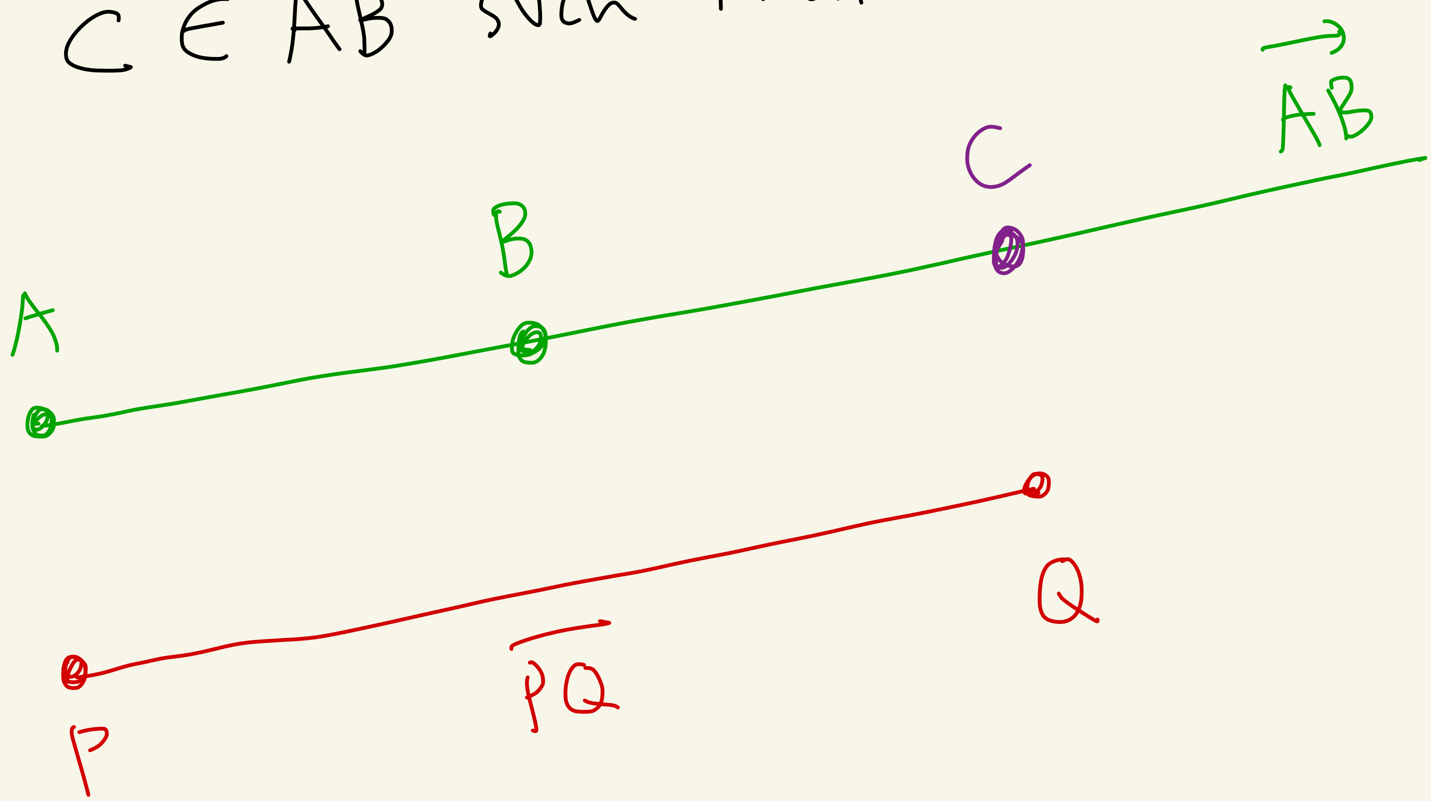

recall this means $d(A,B) = d(C,D)$
ie the length of $\overline{AB}$ equals the length of $\overline{CD}$.

**Theorem:** (Segment construction Theorem)

Let $(\mathcal{P}, \mathcal{L}, d)$ be a metric geometry. Let $A, B, P, Q \in \mathcal{P}$ with $A \neq B$ and $P \neq Q$.

Consider the ray $\overrightarrow{AB}$ and the line segment $\overline{PQ}$.

There exists a unique point $C \in \overrightarrow{AB}$ such that $\overline{AC} \cong \overline{PQ}$



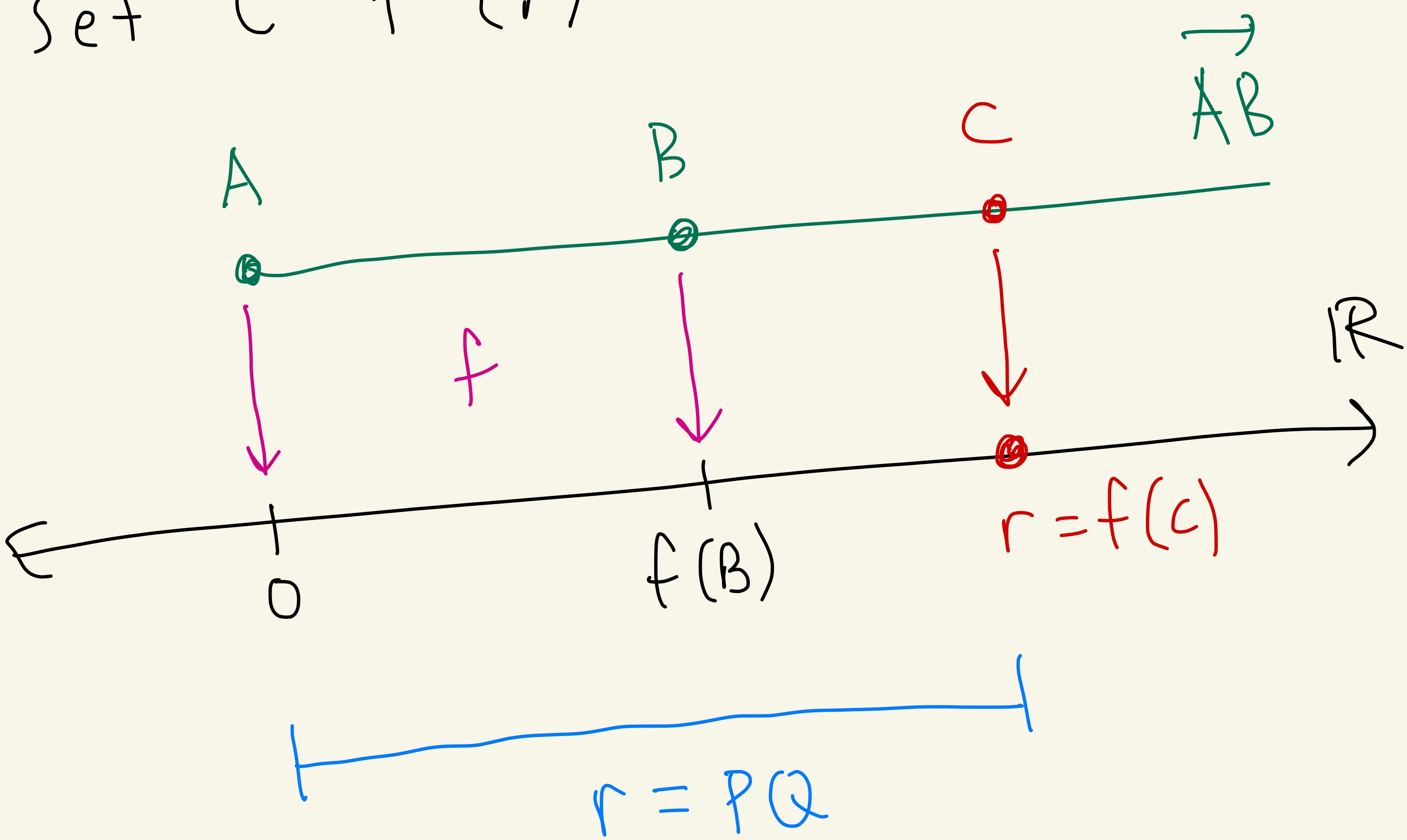
Proof: Let $f: \overleftrightarrow{AB} \to \mathbb{R}$ be a ruler where $f(A)=0$ and $f(B)>0$.

By the previous theorem

$$\overrightarrow{AB} = \{X \in \overleftrightarrow{AB} \mid f(X) \geq 0\}$$

Let $r = PQ$. $\leftarrow$ $\boxed{r = d(P,Q)}$

Set $C = f^{-1}(r)$



Then,

since f is a ruler

$$\begin{aligned} AC &= d(A,C) \\ &= |f(A) - f(C)| \\ &= |0 - r| \\ &= |r| \\ &= r \\ &= PQ \end{aligned}$$

r > 0

So, $\overline{AC} \simeq \overline{PQ}$

Why is C unique?

Suppose there was another point $C' \in \overrightarrow{AB}$ with $\overline{AC'} \simeq \overline{PQ}$.

Since $C' \in \overrightarrow{AB}$ we know $f(C') \geq 0$.

Then,

$$f(C') = f(C') - f(A)$$

since $f(A) = 0$

$$= |f(C') - f(A)|$$

since $f(C') \geq 0 = f(A)$

$$= d(C', A)$$

f is a ruler

$$= d(A, C')$$

$$= AC'$$

$$= PQ$$

$$= r$$

$$= f(C)$$

Since $f$ is 1-1 and $f(C') = f(C)$
we get $C = C'$.

So, $C$ is the unique point on
$\overrightarrow{AB}$ where $\overline{AC} \simeq \overline{PQ}$. $\blacksquare$