Math 4300 Homework 1 Solutions

1 (a) $P=(-1,2)$, $Q=(3,2)$

They don't lie on a vertical line, so plug them into $y=mx+b$ to get:

$$\boxed{\begin{aligned} 2 &= m(-1)+b \\ 2 &= m(3)+b \end{aligned}}$$

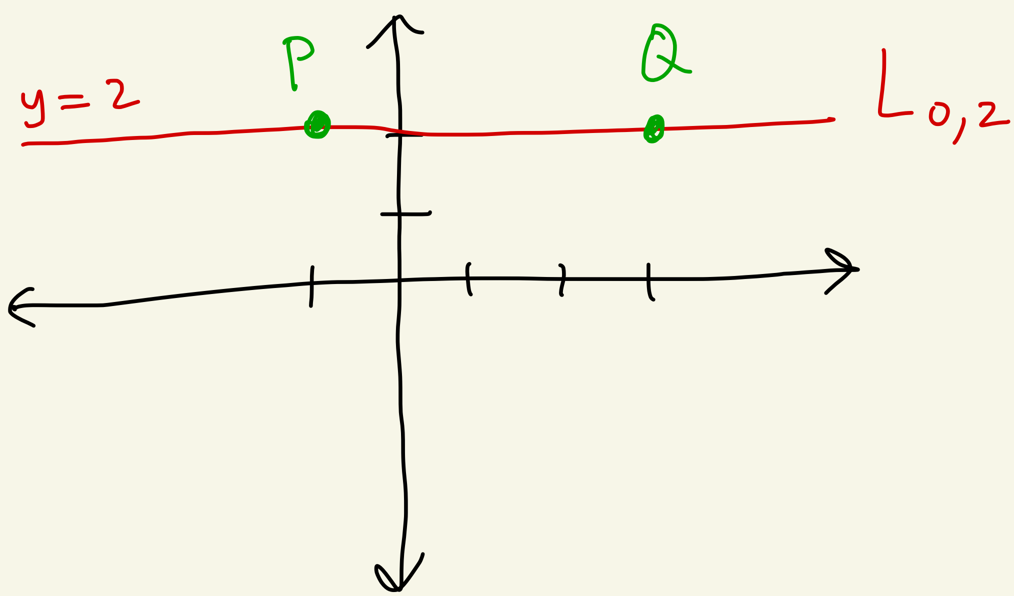
or

$$\boxed{\begin{aligned} -m+b &= 2 \quad (1) \\ 3m+b &= 2 \quad (2) \end{aligned}}$$

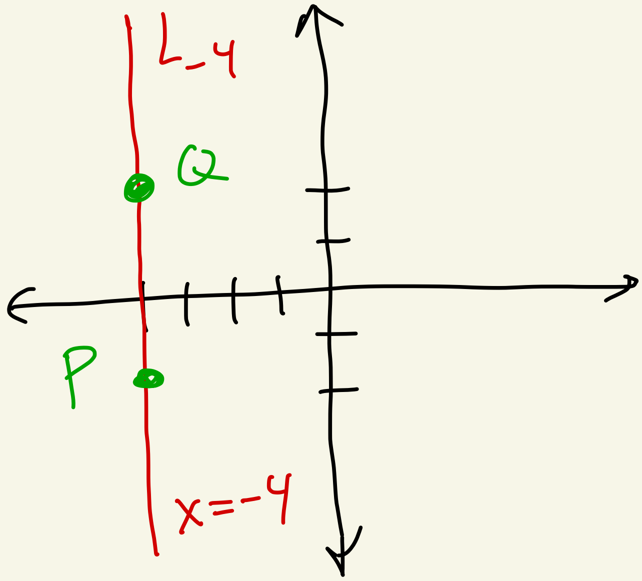
(1) − (2) gives $-4m=0$ or $m=0$.

Thus, $b=2$.

Thus, P, Q lie on $L_{m,b}=L_{0,2}$.



① (b) $P = (-4, -\sqrt{2})$, $Q = (-4, 2)$ have the same x-coordinate and thus lie on the vertical line L_{-4}.



① (c) $P = (2,1)$, $Q = (4,3)$

They don't lie on a vertical line so they must lie on $y = mx+b$ for some m, b.

Plug them into $y = mx+b$ to get:

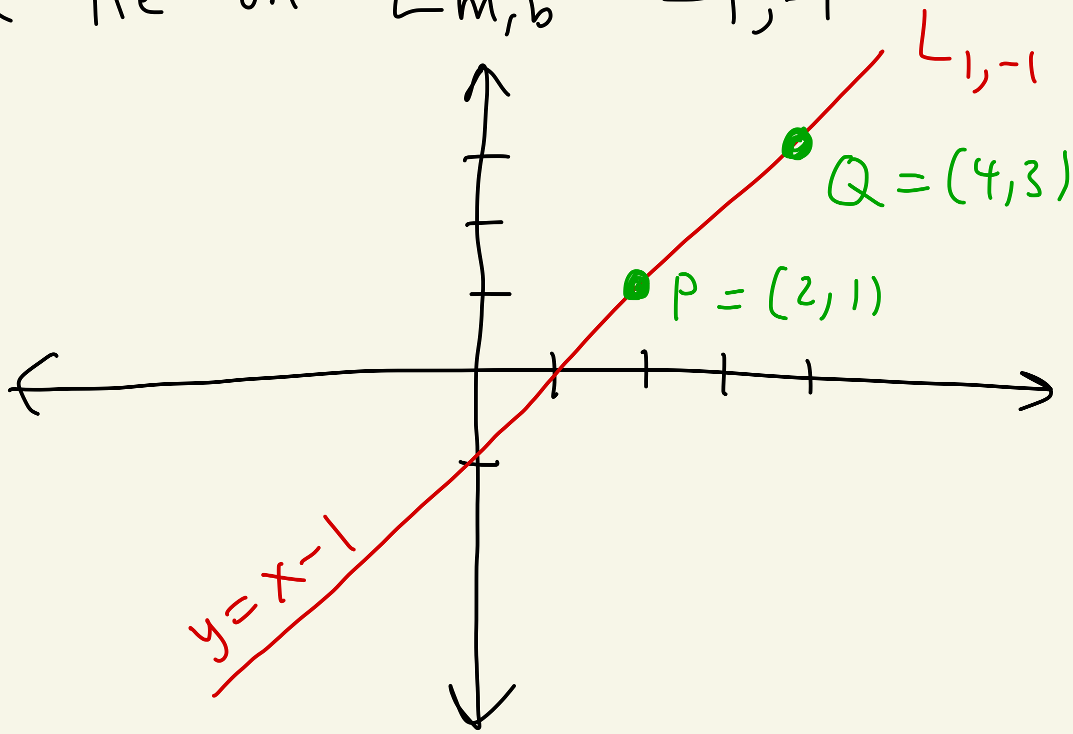
$$\boxed{\begin{aligned} 1 &= m(2)+b \quad ① \\ 3 &= m(4)+b \quad ② \end{aligned}}$$

① − ② gives $-2 = -2m$. So, $m = 1$.

Thus, $b = 1 - 2m = 1 - 2 = -1$.

Therefore,

P, Q lie on $L_{m,b} = L_{1,-1}$



$\boxed{(2)\ (a)}$ $P=(1,2)$, $Q=(3,4)$

P, Q don't lie on a vertical line so they must lie on some ${}_cL_r$.

Plugging them into $(x-c)^2+y^2=r^2$ gives:

$$\boxed{\begin{aligned}(1-c)^2+2^2&=r^2\\(3-c)^2+4^2&=r^2\end{aligned}}$$

which becomes

$$\boxed{\begin{aligned}c^2-2c+5&=r^2 \quad (1)\\c^2-6c+25&=r^2 \quad (2)\end{aligned}}$$

(1) − (2) gives $4c-20=0$.

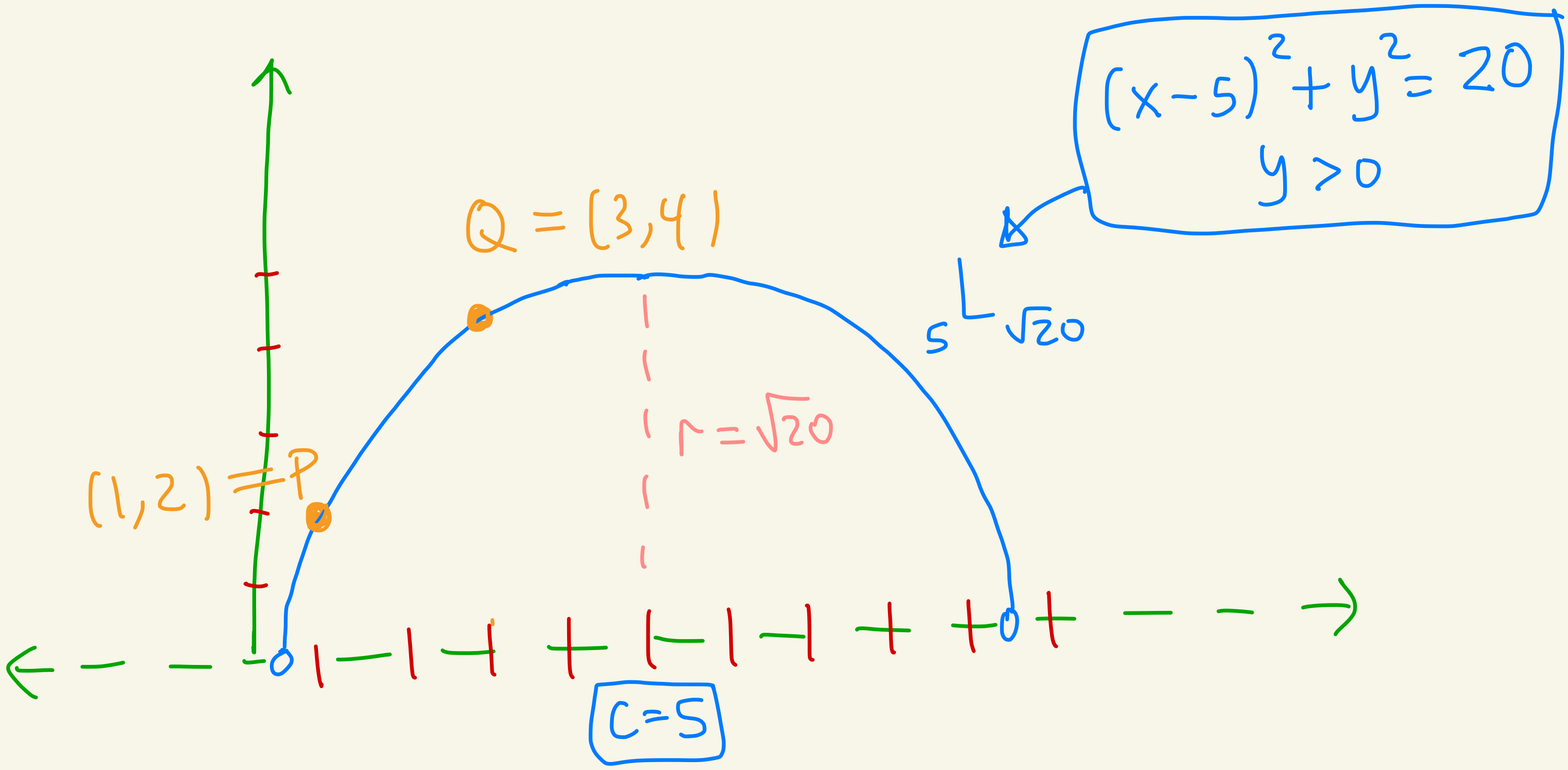
So, $c=5$.

Then, (1) gives $r^2=5^2-2(5)+5=20$.

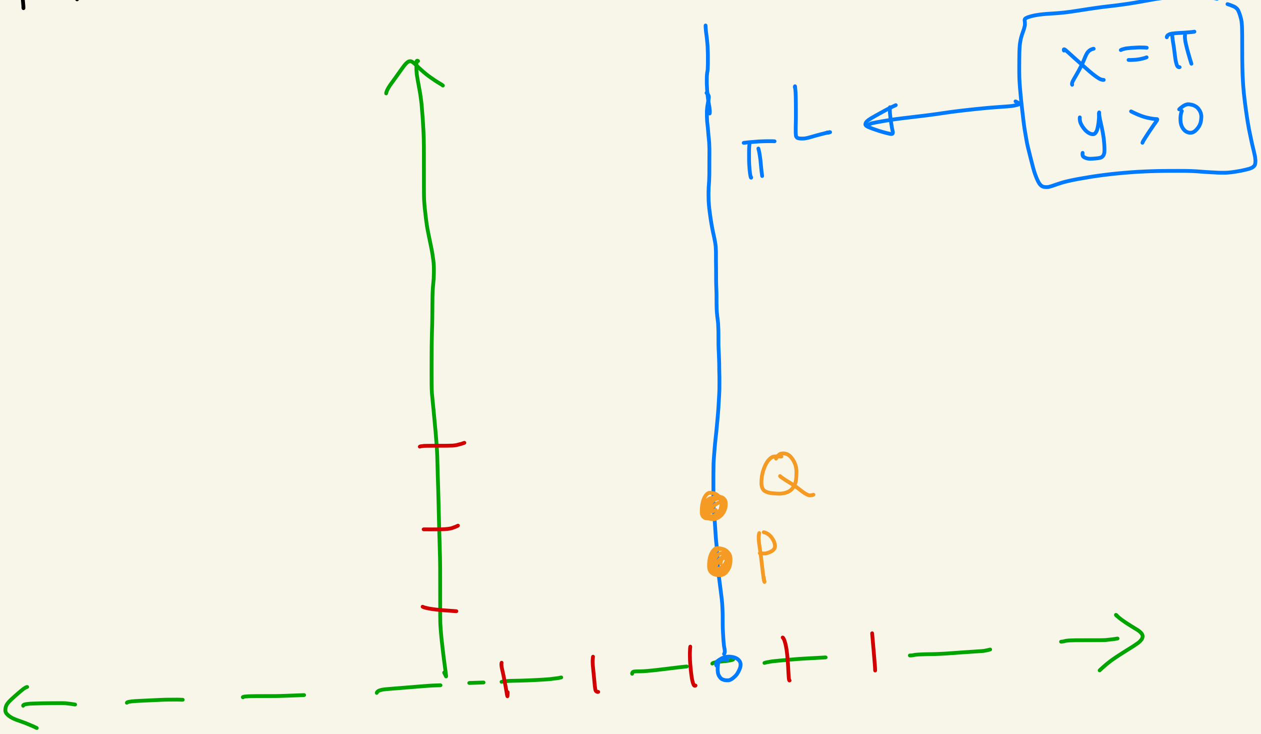
So, $r=\sqrt{20}\approx 4.47$.

Thus, $P=(1,2)$ and $Q=(3,4)$ lie on ${}_5L_{\sqrt{20}}$.

Picture ↴



(2)(b) $P=(\pi,\sqrt{2})$, $Q=(\pi,2)$ lie on the vertical line ${}_\pi L$.



$\boxed{②(c)}$ $P = (2,1)$, $Q = (4,3)$

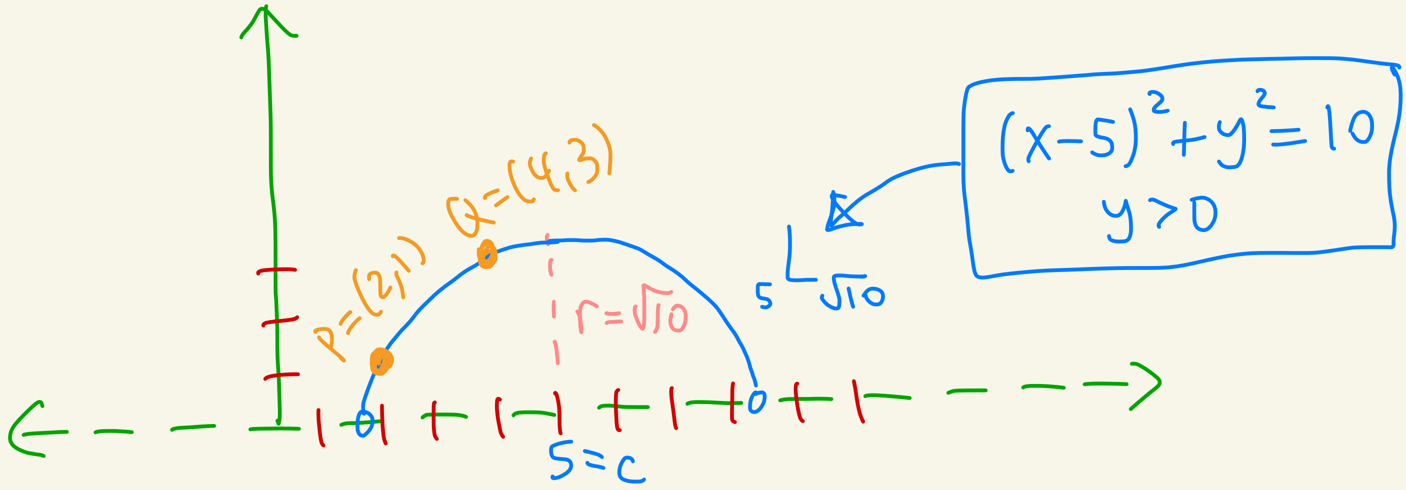
These points don't lie on a vertical line.
So, they must lie on ${}_cL_r$ for some c, r.
plug P, Q into $(x-c)^2 + y^2 = r^2$ to get;

$$\boxed{\begin{aligned}(2-c)^2 + 1^2 &= r^2 \\ (4-c)^2 + 3^2 &= r^2\end{aligned}}$$

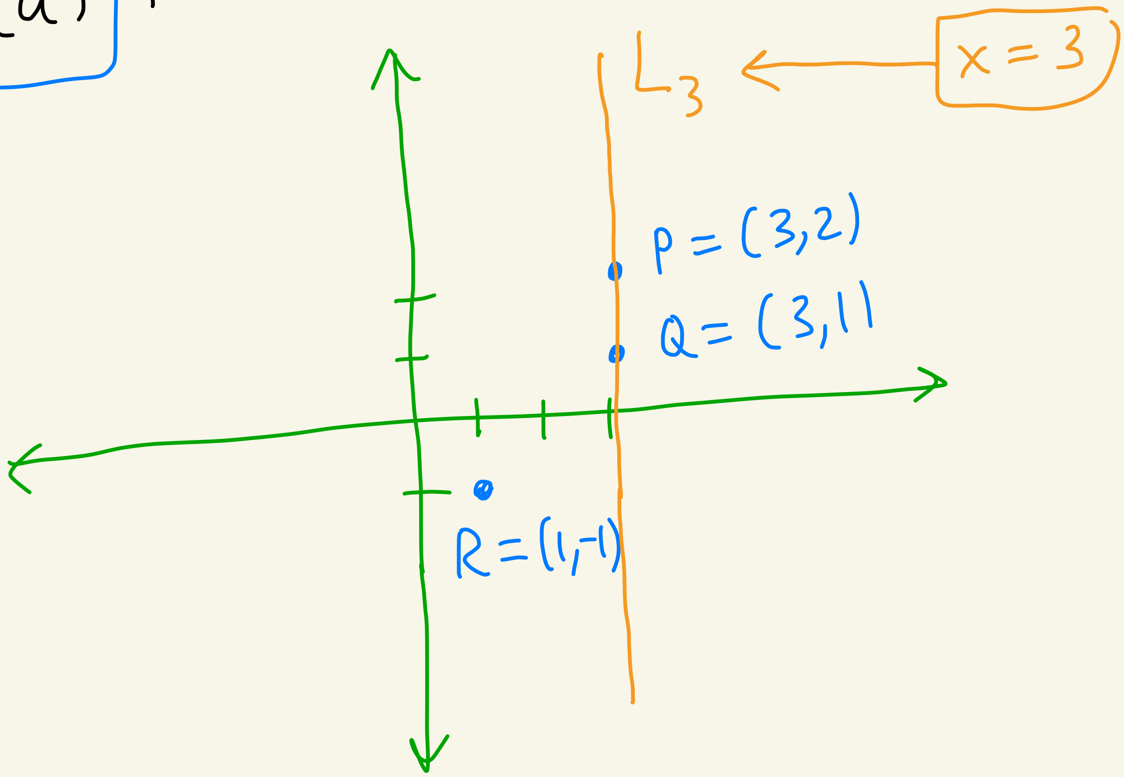
This gives

$$\boxed{\begin{aligned}c^2 - 4c + 5 &= r^2 \\ c^2 - 8c + 25 &= r^2\end{aligned}} \quad \begin{matrix} ① \\ ② \end{matrix}$$

① − ② gives $4c - 20 = 0$.
So, $c = 5$.
① then gives $r^2 = 5^2 - 4(5) + 5 = 10$.
So, $r = \sqrt{10} \approx 3.16$



$\boxed{③ (a)}$ $P=(3,2)$, $Q=(3,1)$, $R=(1,-1)$.



Suppose P, Q, R were collinear. Then they would all lie on the same unique line ℓ. (since a line through any two points in unique)

Since P and Q both lie on L_3

this would imply that $\ell = L_3$.

But $R=(1,-1)$ does not satisfy $x=3$.

So, there is no unique line that P, Q, R all lie on.

Thus, P, Q, R are noncollinear.

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3 (b) Let $P=(2,1)$, $Q=(4,3)$, $R=(6,5)$

Are these points collinear?

Suppose they all lie on some line ℓ.

ℓ can't be vertical since P, Q, R have different x-components.

So ℓ must be of the form $y=mx+b$.

Plugging $P=(2,1)$ and $Q=(4,3)$ into $y=mx+b$ gives

$$\boxed{\begin{aligned} 1 &= m(2)+b \\ 3 &= m(4)+b \end{aligned}} \Longrightarrow \boxed{\begin{aligned} 2m+b &= 1 \quad (1) \\ 4m+b &= 3 \quad (2) \end{aligned}}$$

Doing (1) − (2) gives $-2=-2m$.

Thus, $m=1$.

And by ① $b = 1-2m = 1-2 = -1$.

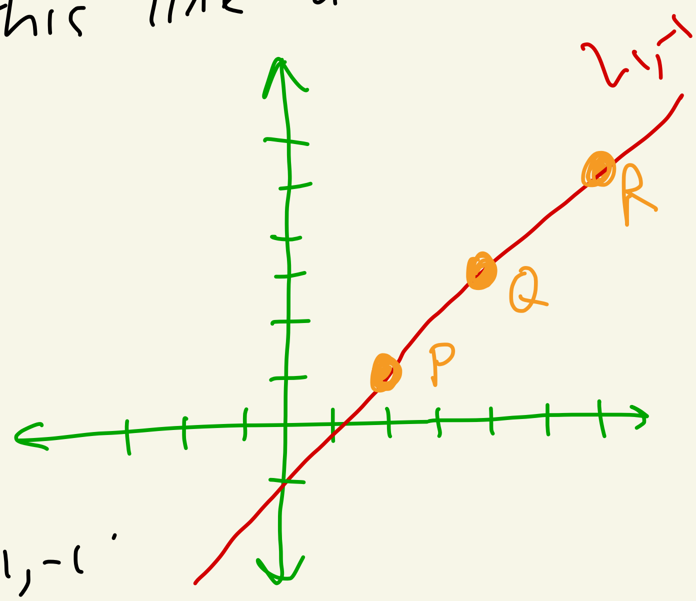
So, P and Q both lie on $L_{m,b} = L_{1,-1}$

$y = x-1$

Since the line through P and Q is unique we have $\ell = L_{1,-1}$.

Does $R=(6,5)$ lie on this line also?

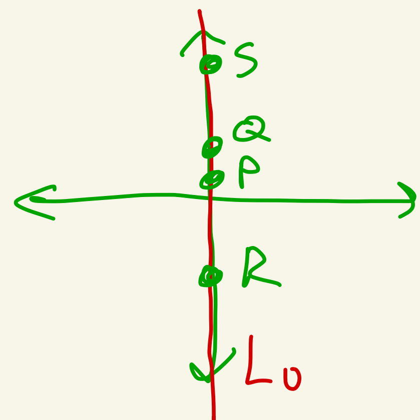
We have $\underbrace{5 = 6-1}_{y=x-1 \text{ with } x=6,\ y=5}$.



So, R also lies on $L_{1,-1}$.

Thus, P, Q, R are collinear.

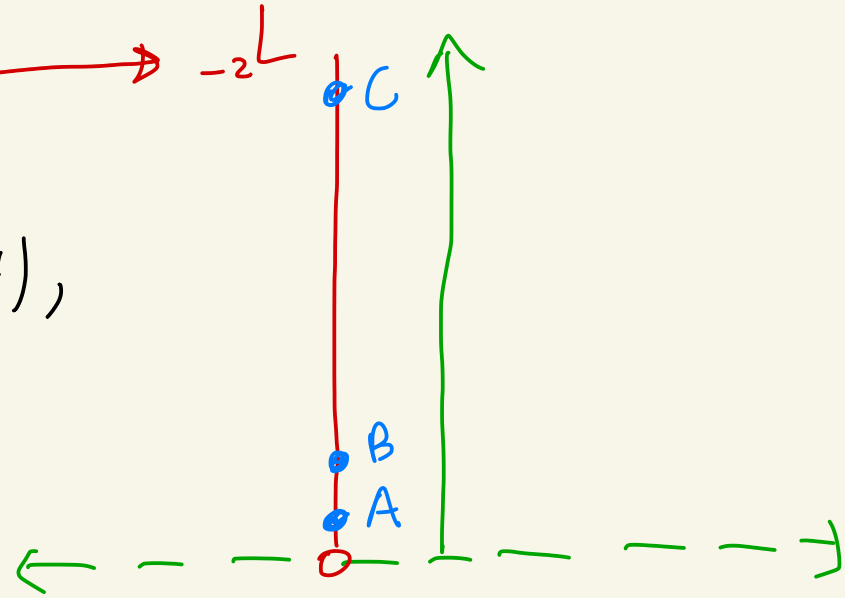
$\boxed{③(c)}$ $P=(0,1)$, $Q=(0,3)$, $R=(0,-5)$, $S=(0,10)$ are collinear since they all lie on $x=0$, ie L_0.



4 (a)

$x=-2$
$y>0$

$A=(-2,2)$, $B=(-2,4)$, $C=(-2,300)$ all lie on ${}_{-2}L$.



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4 (b) $P=(0,1)$, $Q=(1,2)$, $R=(4,1)$

Are these points collinear?

Suppose they are.

Then they lie on a unique line ℓ.

Since they have different x coordinates the line is not vertical, and is of the form $\ell = {}_{c}L_{r}$

$(x-c)^2+y^2=r^2$
$y>0$

Plugging P and Q into $(x-c)^2+y^2=r^2$ and solving gives

$$(0-c)^2+1^2=r^2$$
$$(1-c)^2+2^2=r^2$$

$$\boxed{\begin{aligned} c^2+1 &= r^2 \quad (1) \\ c^2-2c+5 &= r^2 \quad (2) \end{aligned}}$$

$$\boxed{\begin{aligned} &(x-2)^2+y^2=5 \\ &y>0 \end{aligned}}$$

(1) − (2) gives $2c-4=0$. So, $c=2$.

Then, $r^2 = c^2+1 = 2^2+1 = 5$

So, $r = \sqrt{5} \approx 2.236$

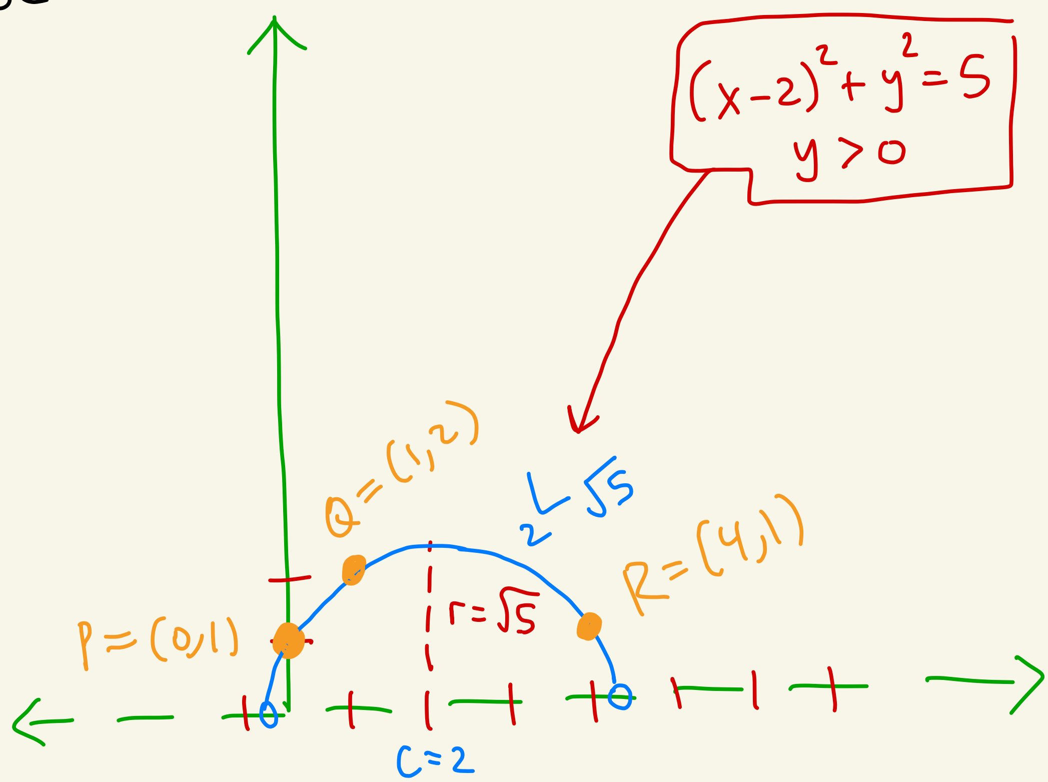
Thus, $P=(0,1)$, and $Q=(1,2)$ lie on ${}_2L_{\sqrt{5}}$

Does $R=(4,1)$ lie on ${}_2L_{\sqrt{5}}$?

If $x=4, y=1$, then $(x-2)^2+y^2=(4-2)^2+1^2=5$

So, yes R does.

Thus, P, Q, R are all collinear.



4 (c) $A=(1,1)$, $B=(3,1)$, $C=(2,3)$

Are these points collinear?

Suppose they are.

Then they all lie on some unique line ℓ.

Since they have different x-coordinates they must all lie on $\ell = {}_cL_r$ for some c, r.

Plugging $A=(1,1)$, $B=(3,1)$ into $(x-c)^2+y^2=r^2$ gives

$$(1-c)^2+1^2=r^2$$
$$(3-c)^2+1^2=r^2$$

$$c^2-2c+2=r^2 \quad (1)$$
$$c^2-6c+10=r^2 \quad (2)$$

(1) − (2) gives $4c-8=0$.

So, $c=2$.

Then, (1) gives $r^2 = 2^2-2(2)+2=2$.

So, $r=\sqrt{2}\approx 1.414$

Thus, $A=(1,1)$, $B=(3,1)$ lie on ${}_cL_r = {}_2L_2$.

$$\boxed{\begin{array}{c}(x-2)^2+y^2=4 \\ y>0\end{array}}$$

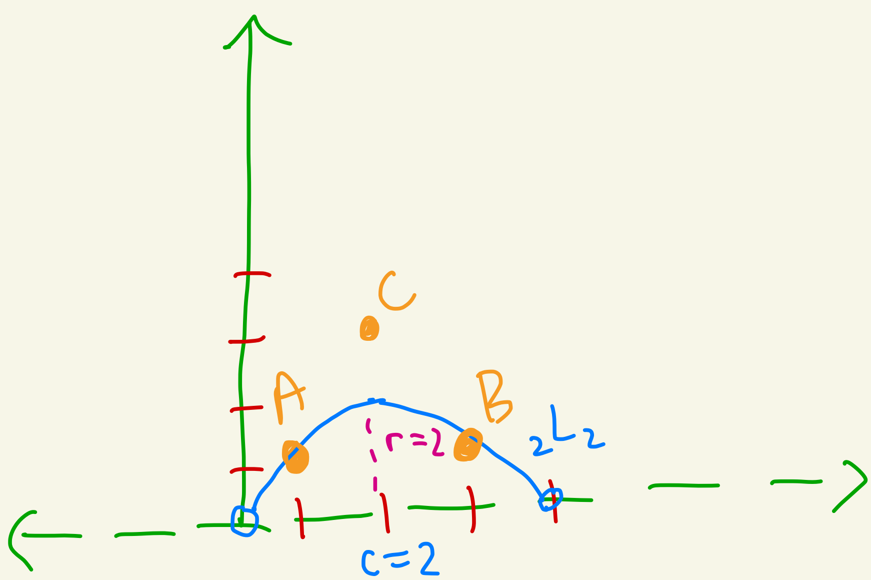
Does $C=(2,3)$ also lie on this line?

Plugging $x=2, y=3$ in we get

$$(x-2)^2+y^2=(2-2)^2+3^2=9\neq 4$$

So, C does not lie on ${}_2L_2$.

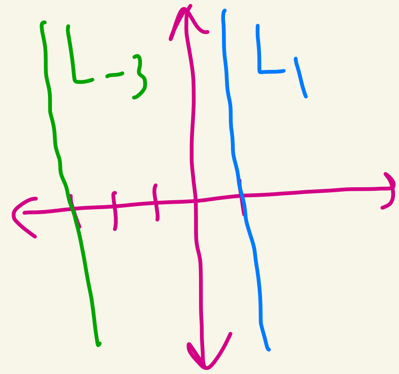
Thus, there is no unique line that passes through A, B, C and these points are noncollinear.

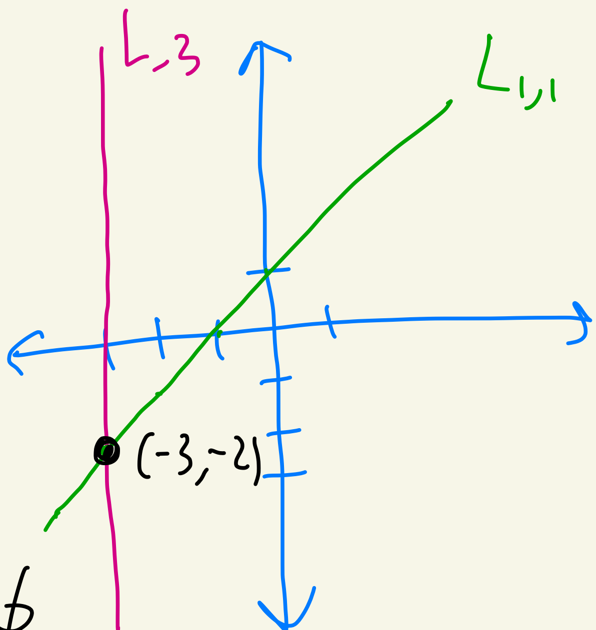


$\boxed{⑤(a)}$ $L_1 = L_1$ so they are parallel

$\boxed{⑤(b)}$ $L_{-3} \neq L_1$ and $L_{-3} \cap L_1 = \phi$.

So, L_{-3} and L_1 are parallel.





$\boxed{⑤(c)}$ $L_{-3} \neq L_{1,1}$

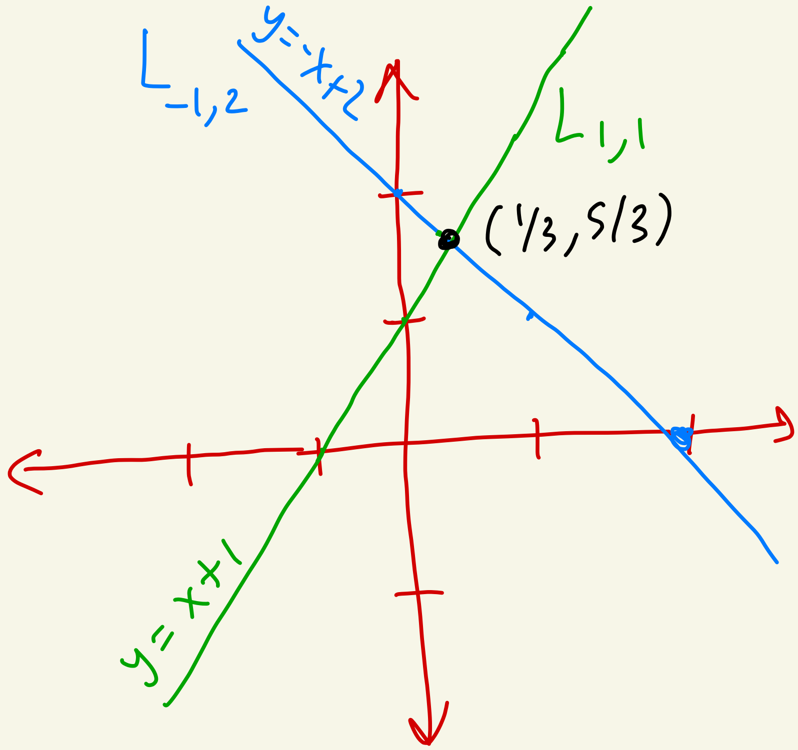
Do they intersect?

Plug $x = -3$ into $y = x + 1$ to get $y = -3 + 1 = -2$.

So, $L_{-3} \cap L_{1,1} = \{(-3,-2)\} \neq \phi$

Thus, L_{-3} and $L_{1,1}$ are not parallel.

5(d) $L_{-1,2} \nparallel L_{1,1}$



Do they intersect.

Take $y=-x+2$ and plug it into $y=x+1$ to get $-x+2=x+1$

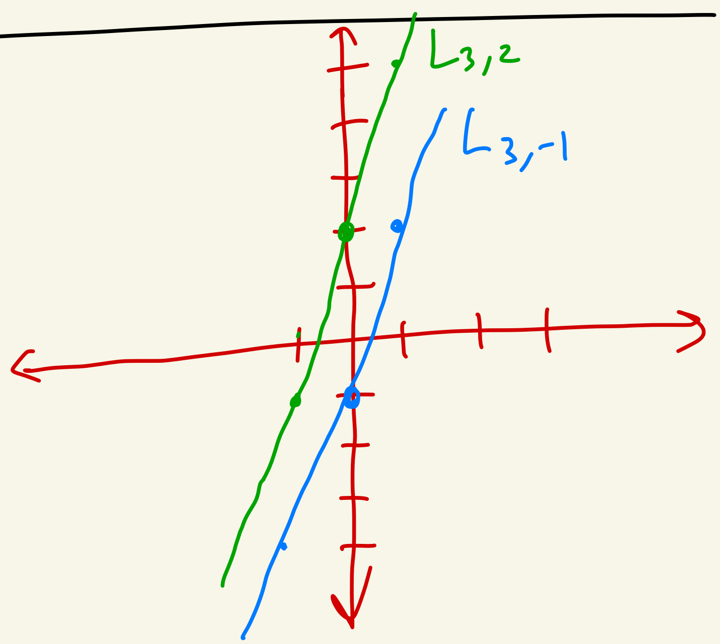
Then, $x=1/3$.

Plug this into $y=-x+2$ to get $y=-1/3+2$
$= 5/3$.

Thus, $L_{-1,2} \cap L_{1,1} = \{(\frac{1}{3}, \frac{5}{3})\} \neq \phi$

and so the lines are not parallel.

5(e) $L_{3,2} \nparallel L_{3,-1}$



Plugging $y=3x+2$ into $y=3x-1$ gives $3x+2=3x-1$.

This gives $2=-1$ which can't be solved.

So, $L_{3,2} \cap L_{3,-1} = \phi$.

Thus, $L_{3,2}$ and $L_{3,-1}$ are parallel.

(6)(a) ${}_0L_1 \neq {}_5L_2$

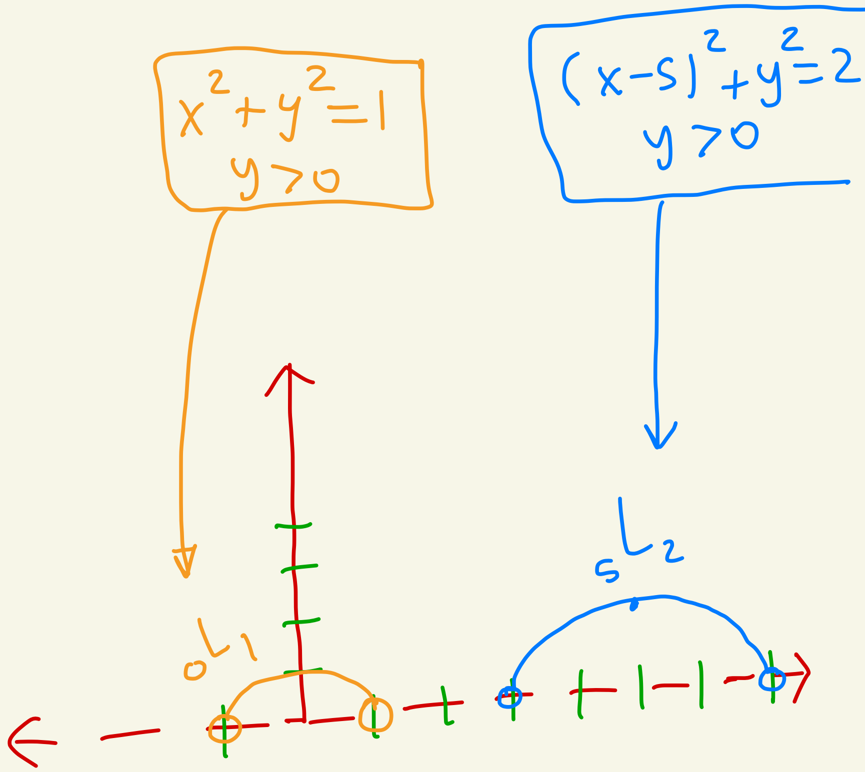
Do they intersect?

We have two equations:

$$\boxed{\begin{array}{l} (x-0)^2+y^2=1 \\ (x-5)^2+y^2=2 \end{array}} \begin{array}{l} \leftarrow {}_0L_1 \\ \leftarrow {}_5L_2 \end{array}$$

$$\Downarrow$$

$$\boxed{\begin{array}{l} x^2+y^2=1 \\ x^2-10x+25+y^2=2 \end{array}} \begin{array}{l} (1) \\ (2) \end{array}$$



(1) - (2) gives $10x-25=-1$.

That gives $10x=24$.

That gives $x=\frac{24}{10}=\frac{12}{5}$.

Plug $x=\frac{12}{5}$ into (1) to get

$$y^2=1-x^2=1-\frac{12^2}{5^2}=\frac{-119}{25}$$

But there is no y with

$$y^2=\frac{-119}{25}.$$

Thus, ${}_0L_1 \cap {}_5L_2=\phi$.

So, ${}_0L_1$ and ${}_5L_2$ are parallel.

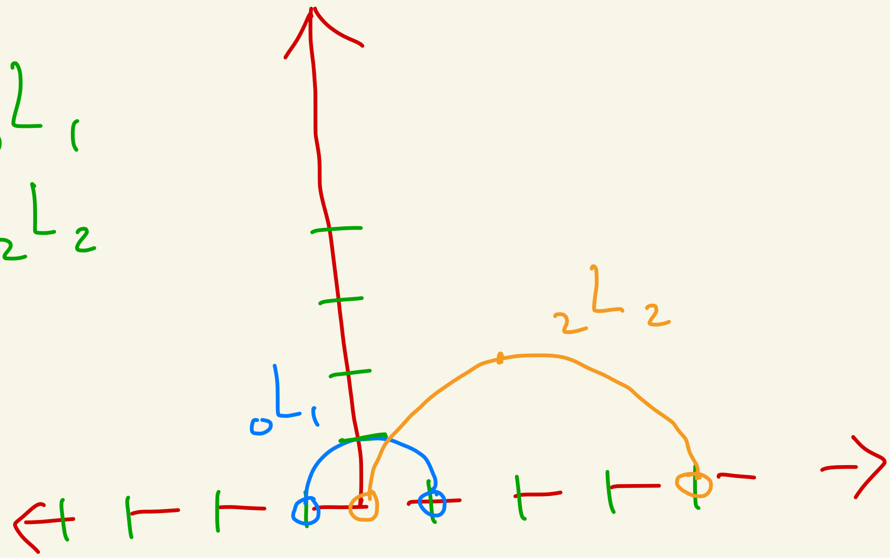
⑥(b)

${}_0L_1 \neq {}_2L_2$

Do they intersect?

We have

$$(x-0)^2 + y^2 = 1^2 \leftarrow {}_0L_1$$
$$(x-2)^2 + y^2 = 2^2 \leftarrow {}_2L_2$$



$$\Downarrow$$

$$x^2 + y^2 = 1 \quad (1)$$
$$x^2 - 4x + 4 + y^2 = 4 \quad (2)$$

(1) − (2) gives $4x - 4 = -3$. So, $x = \frac{1}{4}$.

Plug $x = \frac{1}{4}$ into (1) to get $y^2 = 1 - \frac{1}{16} = \frac{15}{16}$.

So, $y = \sqrt{15/16}$.

Similarly $x = \frac{1}{4}$ in (1) gives $y^2 = 2^2 - \left(\frac{1}{4} - 2\right)^2$
$= 4 - \left(\frac{-7}{4}\right)^2 = \frac{64-49}{16}$
$= 15/16$

So, ${}_0L_1 \cap {}_2L_2 = \{(1/4, 15/16)\} \neq \phi$

Thus, ${}_0L_1$ and ${}_2L_2$ are not parallel.

⑥(c) ${}_0L_{10} \neq {}_5L_2$

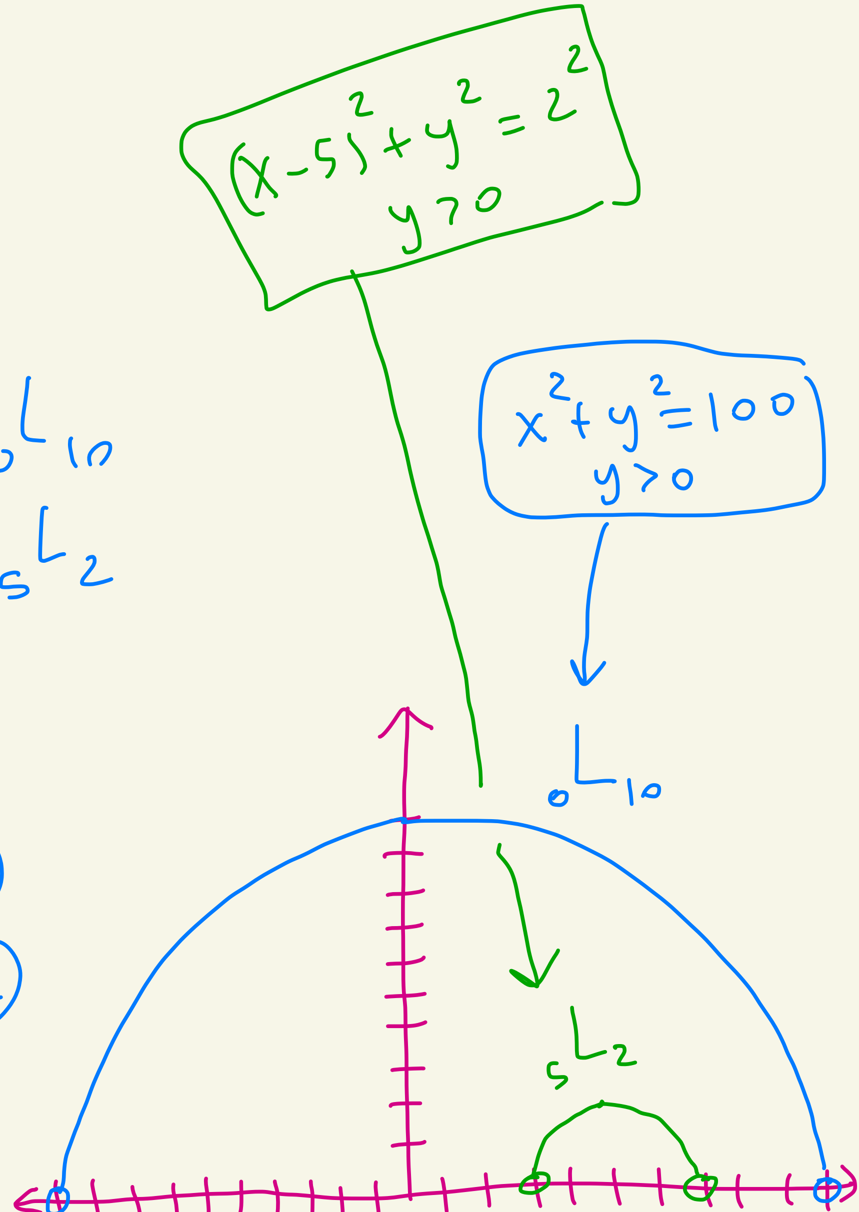
Do they intersect?

We have the equations

$$\boxed{\begin{aligned}(x-0)^2+y^2&=10^2 \quad \leftarrow {}_0L_{10}\\ (x-5)^2+y^2&=2^2 \quad \leftarrow {}_5L_2\end{aligned}}$$

$$\Downarrow$$

$$\boxed{\begin{aligned}x^2+y^2&=100 \quad (1)\\ x^2-10x+25+y^2&=4 \quad (2)\end{aligned}}$$



Then (1) − (2) gives

$$10x-25=96$$
$$10x=121$$
$$x=\frac{121}{10}$$

Plug $x=\frac{121}{10}$ into (1) to get $y^2=100-x^2=100-\frac{14641}{100}$
$$=\frac{-4641}{10}$$

But there is no y with $y^2=\frac{-4641}{10}$.

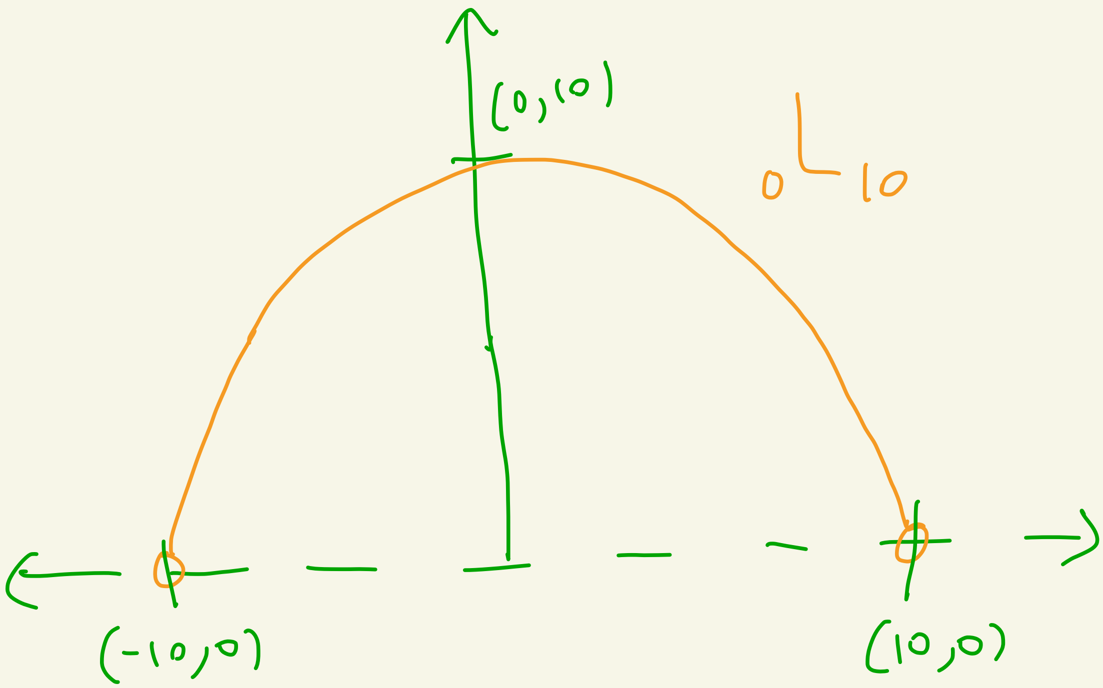
So, ${}_0L_{10}\cap{}_5L_2=\phi$.

Thus, ${}_0L_{10}$ and ${}_5L_2$ are parallel.

⑥ (d)

${}_0L_{10} = {}_0L_{10}$

So, ${}_0L_{10}$ and ${}_0L_{10}$ are parallel.

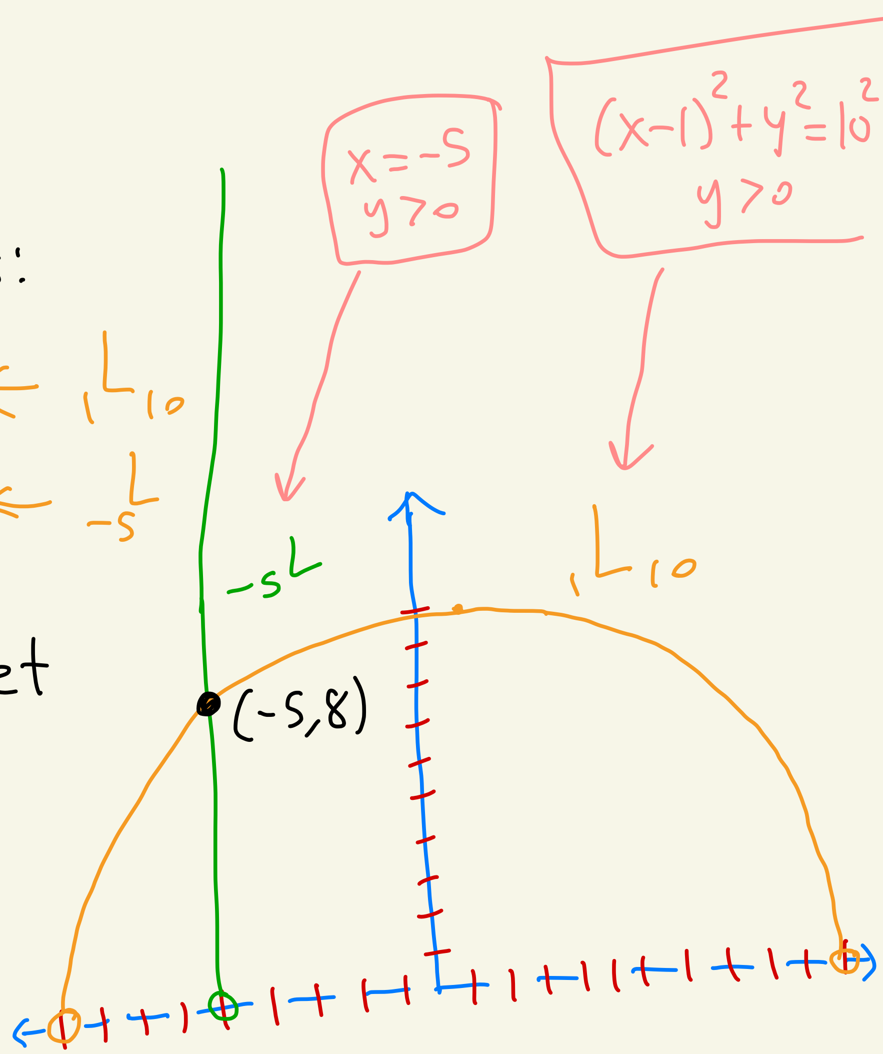


⑥ (e) ${}_1L_{10} \nparallel {}_{-5}L$

Do they intersect?

We have these equations:

$$\boxed{\begin{matrix} (x-1)^2 + y^2 = 10^2 & ① \leftarrow {}_1L_{10} \\ x = -5 & ② \leftarrow {}_{-5}L \end{matrix}}$$



Plug ② into ① to get

$$(-5-1)^2 + y^2 = 100$$

$$36 + y^2 = 100$$

$$y^2 = 64$$

$$y = \sqrt{64} = 8$$

Thus, ${}_1L_{10} \cap {}_{-5}L = \{(-5,8)\} \neq \emptyset$

So, ${}_1L_{10}$ and ${}_{-5}L$ are not parallell.

⑥ (f) ${}_1L_1 \neq {}_2L_2$

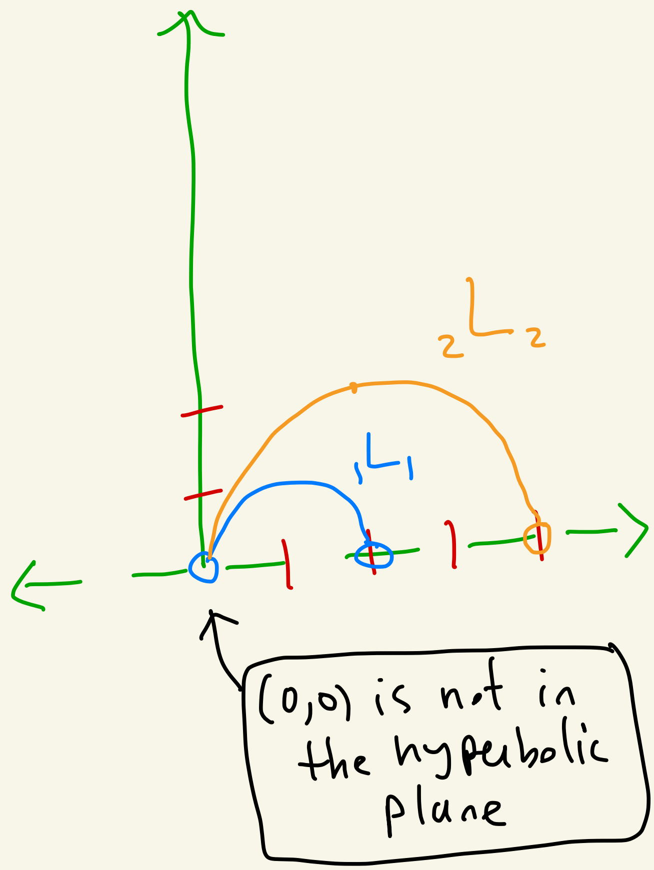
Do they intersect?

We have

$$\boxed{\begin{aligned}(x-1)^2+y^2&=1^2 \quad \leftarrow {}_1L_1\\ (x-2)^2+y^2&=2^2 \quad \leftarrow {}_2L_2\end{aligned}}$$

$\Downarrow$

$$\boxed{\begin{aligned}x^2-2x+1+y^2&=1 \quad (1)\\ x^2-4x+4+y^2&=4 \quad (2)\end{aligned}}$$



(1) − (2) gives $2x-3=-3$.

So, $x=0$.

Plug $x=0$ into (1) to get
$y^2=0$, and so $y=0$.

Plug $x=0$ into (2) to get
$y^2=0$ and so $y=0$.

However, even though $(x,y)=(0,0)$ satisfies (1) and (2) we have $(0,0)$ is not in the hyperbolic plane since its y-coordinate isn't positive. So, ${}_1L_1 \cap {}_2L_2 = \phi$ and these lines are parallell.

⑦ Let $(\mathscr{P}, \mathscr{L})$ be an incidence geometry.

Let P, Q, R be distinct points from $\mathscr{P}$ that are collinear.

Then there exists a line ℓ from $\mathscr{L}$ where P, Q, and R all lie on ℓ.

We must show ℓ is unique.

Since $(\mathscr{P}, \mathscr{L})$ is an incidence geometry there is a unique line through any two distinct points.

That is the only line through P and Q is $\overleftrightarrow{PQ}$.

Thus, $\ell = \overleftrightarrow{PQ}$ and it is unique.

$\boxed{/\!/}$

(8) Let $(\mathscr{P}, \mathscr{L})$ be an incidence geometry.
Let $\ell \in \mathscr{L}$ be a line.
We must show there exists a point $P \in \mathscr{P}$ that does not lie on ℓ.

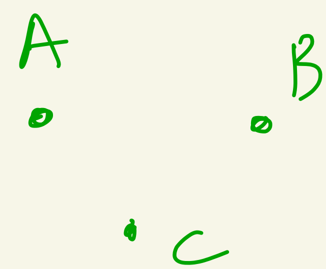
Suppose otherwise.
Then every point $P \in \mathscr{P}$ lies on ℓ.
But then all the points of $\mathscr{P}$ would be collinear.
However, $(\mathscr{P}, \mathscr{L})$ is an incidence plane and thus by definition there must exist three points that are noncollinear.
Contradiction.

Thus, there must exist a point P where P is not on ℓ. $\blacksquare$

9 (Method 1 - proof by contradiction)

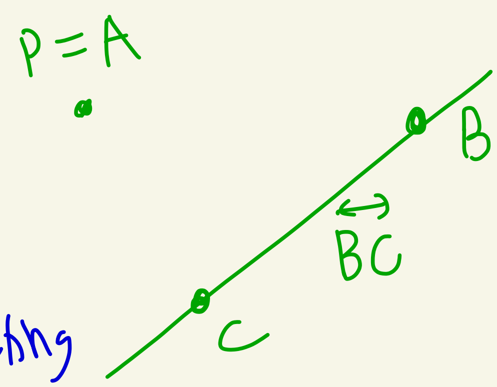
Suppose otherwise.
That is, suppose P lies on every line in $\mathcal{L}$.

Since $(\mathcal{P}, \mathcal{L})$ is an incidence geometry there exists distinct points A, B, C that are non-collinear.



Case 1: Suppose $P \in \{A, B, C\}$.

Without loss of generality, assume $P = A$.
By assumption, $P \in \overleftrightarrow{BC}$
But then $A = P \in \overleftrightarrow{BC}$.
Then, $A, B, C \in \overleftrightarrow{BC}$ contradicting that they are collinear.



Case 2: Suppose $P \notin \{A, B, C\}$.

Claim: Either $P \notin \overleftrightarrow{AB}$ or $P \notin \overleftrightarrow{AC}$, or $P \notin \overleftrightarrow{BC}$.

pf of claim: We just have to rule out the case where $P \in \overleftrightarrow{AB}$, $P \in \overleftrightarrow{AC}$, and $P \in \overleftrightarrow{BC}$.

Suppose $P \in \overleftrightarrow{AB}$ and $P \in \overleftrightarrow{AC}$ and $P \in \overleftrightarrow{BC}$.

Note that $A, P \in \overleftrightarrow{AB}$.

Also, $A, P \in \overleftrightarrow{AC}$.

But there is a unique line through A and P.

Thus, $\overleftrightarrow{AB} = \overleftrightarrow{AC}$.

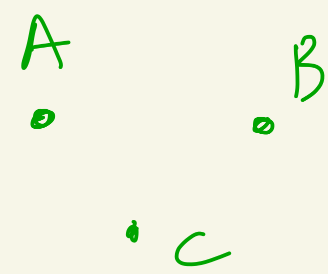
But then $A, B, C \in \overleftrightarrow{AB}$ since $\overleftrightarrow{AB} = \overleftrightarrow{AC}$

which contradicts that A, B, C are non-collinear $\boxed{\text{claim}}$

By case 1 and case 2, there has to be a line that P is not on.

9 (Method 2 - Direct proof)

Since $(\mathscr{P}, \mathscr{L})$ is an incidence geometry there exists distinct points A, B, C that are non-collinear.



Case 1: Suppose $P \in \{A, B, C\}$.

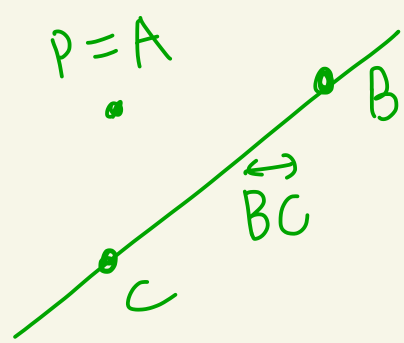
Without loss of generality, assume $P = A$.

Let's show that $P \notin \overleftrightarrow{BC}$.

Suppose $P \in \overleftrightarrow{BC}$.

Then $A = P \in \overleftrightarrow{BC}$.

Then, $A, B, C \in \overleftrightarrow{BC}$ contradizting that they are collinear.



Thus we must have $P \notin \overleftrightarrow{BC}$.

[A similar argument shows that if $P = B$, then $P \notin \overleftrightarrow{AC}$. And if $P = C$, then $P \notin \overleftrightarrow{AB}$]

Case 2: Suppose $P \notin \{A, B, C\}$.

Claim: Either $P \notin \overleftrightarrow{AB}$ or $P \notin \overleftrightarrow{AC}$, or $P \notin \overleftrightarrow{BC}$.

pf of claim: We just have to rule out the case where $P \in \overleftrightarrow{AB}$, $P \in \overleftrightarrow{AC}$, and $P \in \overleftrightarrow{BC}$.

Suppose $P \in \overleftrightarrow{AB}$ and $P \in \overleftrightarrow{AC}$ and $P \in \overleftrightarrow{BC}$.

Note that $A, P \in \overleftrightarrow{AB}$.

Also, $A, P \in \overleftrightarrow{AC}$.

But there is a unique line through A and P.

Thus, $\overleftrightarrow{AB} = \overleftrightarrow{AC}$.

But then $A, B, C \in \overleftrightarrow{AB}$ since $\overleftrightarrow{AB} = \overleftrightarrow{AC}$

which contradicts that A, B, C are non-collinear $\boxed{\text{claim}}$

By case 1 and case 2, there has to be a line that P is not on.

$\blacksquare$

(10) (Method 1)

<u>Proof by contradiction:</u>

Suppose given any two points Q, R we have that P, Q, R are collinear.

Since we have an incidence geometry there must exist distinct points A, B, C that are non-collinear.

<u>Case 1:</u> Suppose $P = A$.

Then, P, B, C are non-collinear which contradicts our assumption.

<u>Case 2:</u> Suppose $P \neq A$.

By assumption, P, A, B are collinear and hence $P, A, B \in \overleftrightarrow{AB}$.

By assumption, P, A, C are collinear and hence $P, A, C \in \overleftrightarrow{AC}$.

Then, $P, A \in \overleftrightarrow{AB}$ and $P, A \in \overleftrightarrow{AC}$.

Since there is a unique line through any two distinct points we know $\overleftrightarrow{AB} = \overleftrightarrow{AC}$

But then $A, B, C \in \overleftrightarrow{AB}$ which contradicts that A, B, C are non-collinear.

Both cases lead to contradictions, so we are done. $\blacksquare$

(10) (Method 2) Let P be some point from $\mathscr{P}$.

By the def of incidence geometry there exist points $A, B, C \in \mathscr{P}$ where A, B, C are non-collinear.

<u>case 1:</u> Suppose P is equal to one of A, B, or C.

For example, suppose $P = A$. Then set, $Q = B$ and $R = C$. We would have then that P, Q, R are non-collinear.

Same idea works if $P = B$ or $P = C$.

<u>case 2:</u> Suppose $P \neq A$, $P \neq B$, and $P \neq C$.

(i) If P, B, C are non-collinear ← (set $Q = B, R = C$)

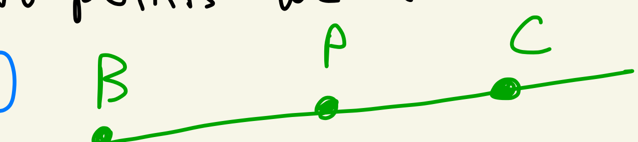
or A, P, C are non-collinear ← [set $Q = A, R = C$]

or A, B, P are non-collinear ← [set $Q = A, R = B$]

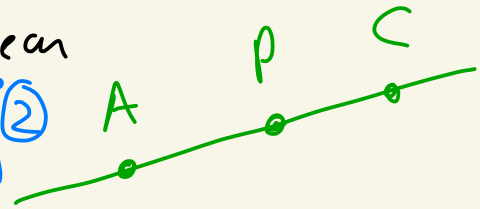
then we are done.

(ii) The only case left is that P, B, C are collinear and A, P, C are collinear, and A, B, P are collinear. We show this can't happen. Suppose it does.

Since P, B, C are collinear and there exists a unique line through any two points we have that $\boxed{\overleftrightarrow{PB} = \overleftrightarrow{PC} = \overleftrightarrow{BC}}$. (1)



Similarly since A, P, C are collinear we have that $\boxed{\overleftrightarrow{AP} = \overleftrightarrow{AC} = \overleftrightarrow{PC}}$ (2)



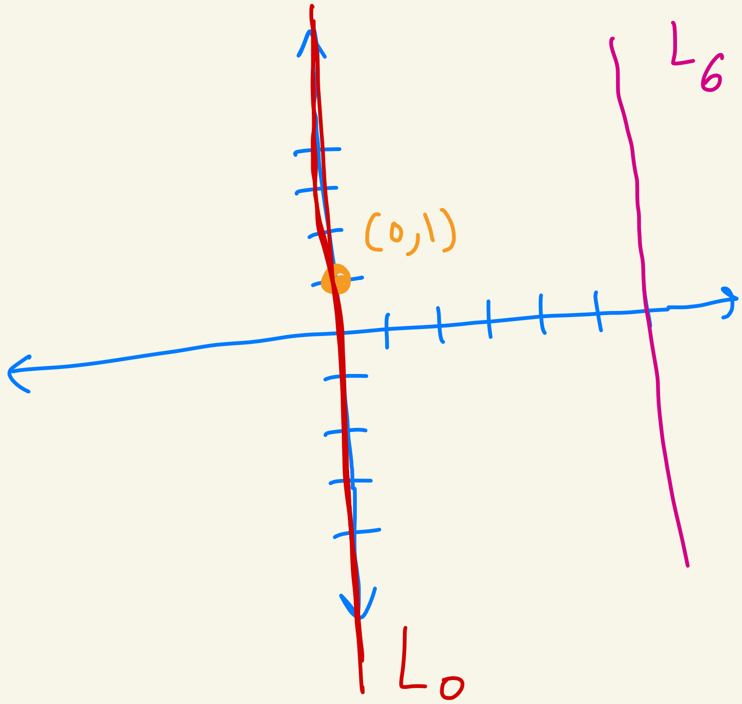
Thus, $\overleftrightarrow{BC} \overset{(1)}{=} \overleftrightarrow{PC} \overset{(2)}{=} \overleftrightarrow{AC}$

But then $A, B, C \in \overleftrightarrow{BC} = \overleftrightarrow{AC}$ contradicting that A, B, C are non-collinear.

Thus, (ii) can't happen.

Thus, by the above there exist $Q, R \in \mathscr{P}$ where P, Q, R are non-collinear. $\blacksquare$

11 (a)



Case (i): Is there a line $\ell = L_a$ through $(0,1)$ that is paralell to L_6? The only line L_a through $(0,1)$ is L_0.

In this case,

$$L_0 \cap L_6 = \{(0,y) \mid y \in \mathbb{R}\} \cap \{(6,y) \mid y \in \mathbb{R}\}$$

$$= \phi.$$

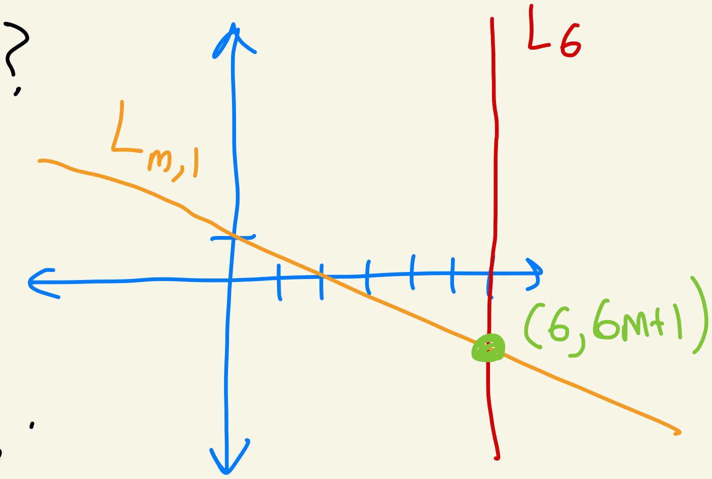
So, L_0 is paralell to L_a

---

case (ii) Suppose $\ell = L_{m,b}$ goes through $(0,1)$.

Then, $\ell = L_{m,1} = \{(x,y) \mid y = mx + 1\}$.

Can ℓ be paralell to L_6?



No.

This is because the point $(6, 6m+1) \in L_{m,1} \cap L_6$.

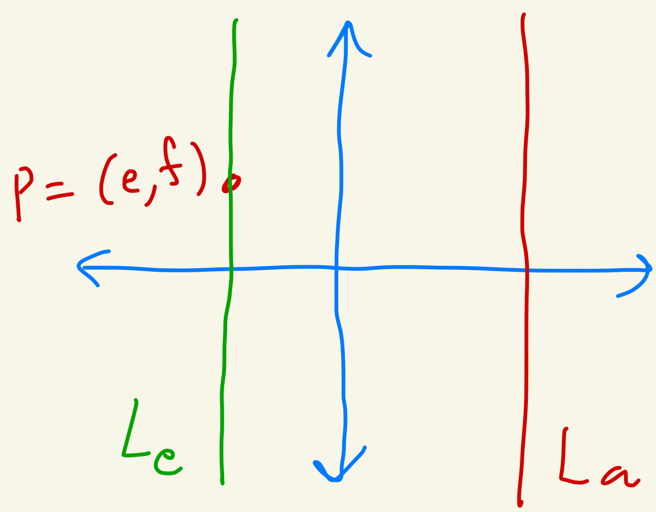
So, $L_{m,1} \cap L_b \neq \phi$ and the two lines are not parallell.

Thus by cases (i) and (ii) we have that the only line through $P = (0,1)$ that is parallell to L_b is L_0. $\blacksquare$

$\boxed{11\,(b)}$

Let ℓ be a line and P a point not on ℓ. We must find a unique line m such that $P \in m$ and m is parallell to ℓ.

$\boxed{\text{Case 1:}}$ Suppose $\ell = L_a$ and $P = (e,f)$ where $P \notin L_a$.



Since $P \notin L_a$ we know $e \neq a$.

Note that $P \in L_e$

And $L_e \cap L_a = \{(e,y) \mid y \in \mathbb{R}\} \cap \{(a,y) \mid y \in \mathbb{R}\}$

$$= \phi \quad \text{(since } e \neq a\text{)}$$

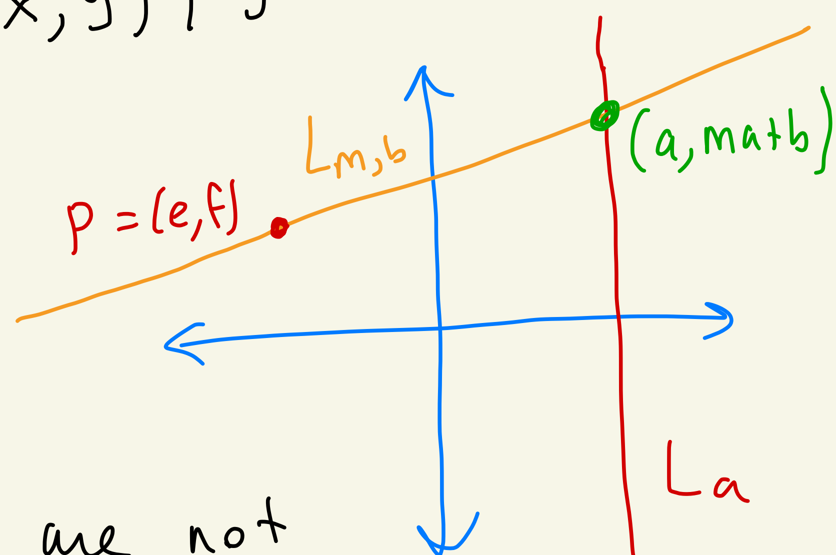
So, $P \in L_e$ and L_e is parallel to L_a.

We show L_e is the only line with this property.

L_e is only vertical line that P lies on.

What about a non-vertical line?

Suppose $P \in L_{m,b} = \{(x,y) \mid y = mx+b\}$



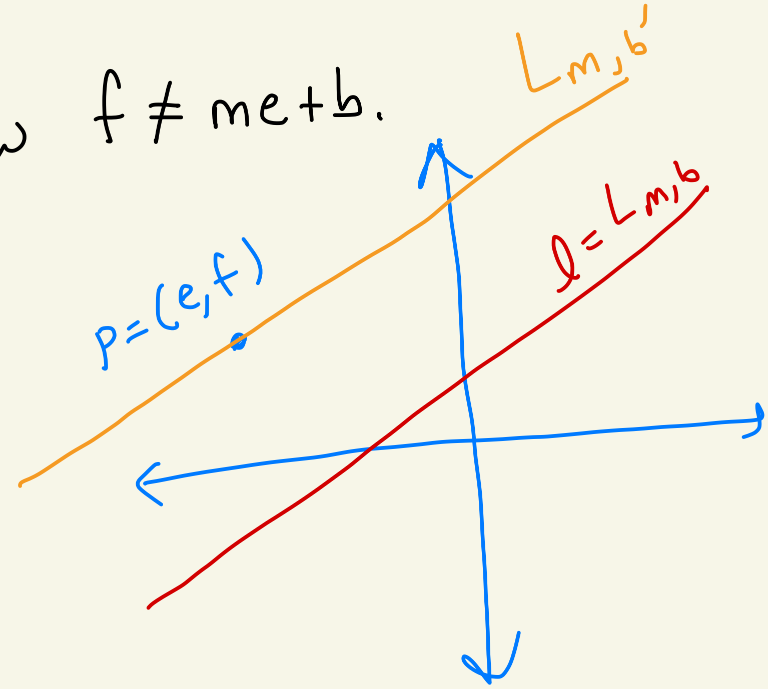
Then,

$(a, ma+b) \in L_{m,b} \cap L_a$

and so $L_{m,b}$ and L_a are not parallell.

$\boxed{\text{Case 2:}}$ Suppose $\ell = L_{m,b}$ and $P = (e,f)$ where $P \notin L_{m,b}$.

Since $P \notin L_{m,b}$ we know $f \neq me + b$.



Consider the line

$$L_{m,b'} = \{(x,y) \mid y = mx + b'\}$$

where $b' = f - me$.

Then, $f = me + b'$ and so

$$p = (e,f) \in L_{m,b'}.$$

Since $b \neq f - me$ and $b' = f - me$ we have that $b \neq b'$.

Thus, $L_{m,b} \cap L_{m,b'} = \phi$ because if

$$(x,y) \in L_{m,b} \cap L_{m,b'}$$

then $\underbrace{mx + b = y}_{(x,y) \in L_{m,b}} \underbrace{= mx + b'}_{(x,y) \in L_{m,b'}}$, but $b \neq b'$.

So, $L_{m,b}$ and $L_{m,b'}$ are parallell.

Thus, $P \in L_{m,b'}$ we $L_{m,b'}$ is parallell to $L_{m,b}$.

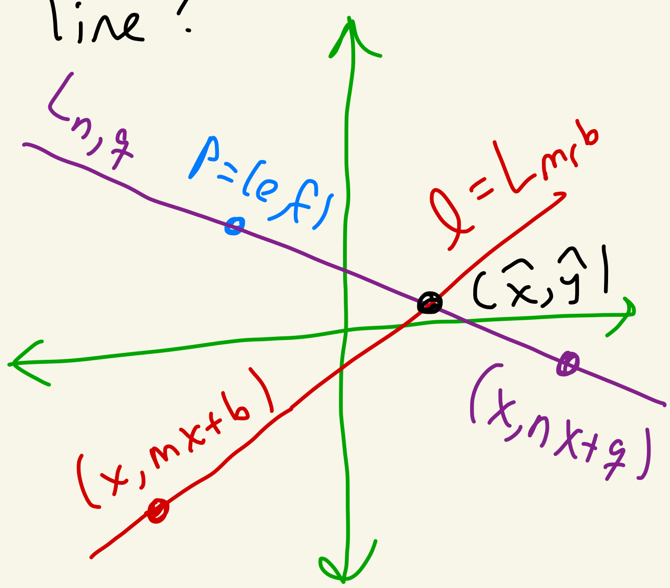
Can there be any other such lines?

Any vertical line L_a must intersect $L_{m,b}$ because $(a, ma+b) \in L_a \cap L_{m,b}$.

What about a non-vertical line?

Suppose $P \in \underbrace{L_{n,q}}_{\text{points that solve } y=nx+q} \neq L_{m,b'}$.

Then, since $P=(e,f) \in L_{n,q}$ we have $f = ne+q$.



If $n=m$, then since $L_{n,q} \neq L_{m,b'}$ we must have $q \neq b'$.

But then $q = f - ne = f - me = b'$.

But that gives both, $\boxed{m=n}$ $q \neq b'$ and $q = b'$.

So we can't have $n=m$.

If $n \neq m$, then by solving $nx+q = y = mx+b$ we get $x = \dfrac{b-q}{n-m}$.

$\boxed{\text{defined since } n \neq m \text{ so } n-m \neq 0}$

We then have $(\hat{x}, \hat{y}) = \left(\frac{b-q}{n-m}, n\left(\frac{b-q}{n-m}\right) + q \right)$

lies on both $L_{n,q}$ and $L_{m,b}$.

This shows $L_{n,q}$ and $L_{m,b}'$ are not parallel.

Why does $(\hat{x}, \hat{y})$ lie on both lines?

Plug it in!

We have

$$\hat{y} - (n\hat{x} + q) = n\left(\frac{b-q}{n-m}\right) + q - \left(n\left(\frac{b-q}{n-m}\right) + q\right) = 0$$

So, $\hat{y} = n\hat{x} + q$.

So, $(\hat{x}, \hat{y}) \in L_{n,q}$.

Also,

$$\hat{y} - (m\hat{x} + b) = n\left(\frac{b-q}{n-m}\right) + q - \left(m\left(\frac{b-q}{n-m}\right) + b\right)$$

$$= \frac{n(b-q) - m(b-q)}{n-m} + q - b$$

$$= \frac{(n-m)(b-q)}{n-m} + (q-b)$$

$$= (b-q) + (q-b) = 0$$

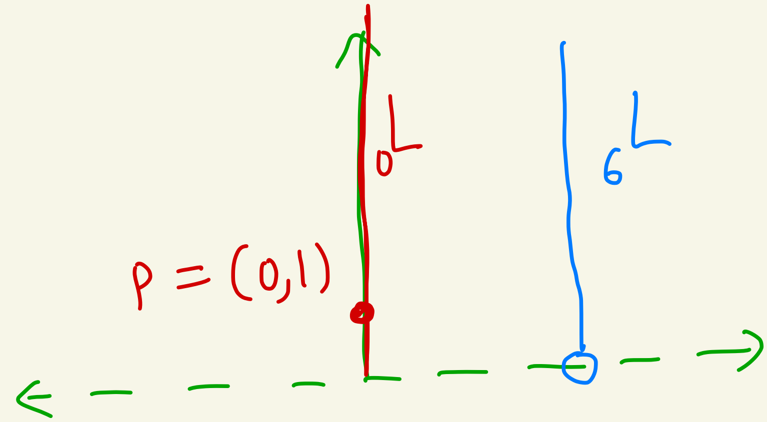
So, $(\hat{x}, \hat{y}) \in L_{m,b}$.

Thus, $L_{n,q} \cap L_{m,b} \neq \phi$ and they aren't parallel.

Therefore, the only line through P that is parallell to $L_{m,b}$ is $L_{m,b'}$.

$\blacksquare$

12 (a)



Let $P=(0,1)$.

Note that $P \in {}_0L$ and ${}_0L \cap {}_6L = \phi$.

So, $P \in {}_0L$ and ${}_0L$ is paralell to ${}_6L$.

Note that ${}_0L$ is the only vertical line that P lives on.

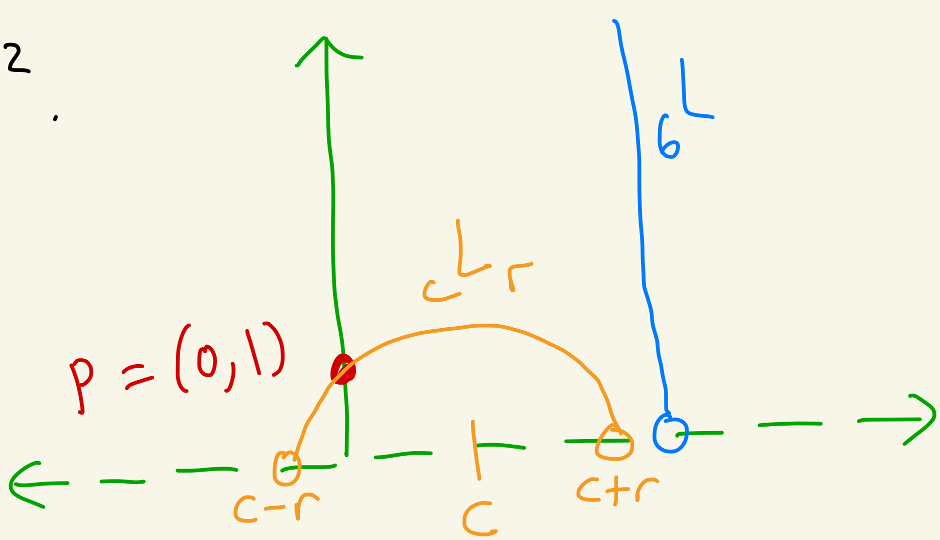
Can we find and ${}_cL_r$ lines that P lives on that are paralell to ${}_6L$?

Suppose $P=(0,1)$ satisfies $(x-c)^2+y^2=r^2$.

Then, $(0-c)^2+1^2=r^2$.

So, $c^2+1^2=r^2$.

Let's consider $c>0$ to start.



We need $c < 6$ and $c + r \leq 6$

so that ${}_cL_r \cap {}_6L = \phi$

For example, pick any c with

$0 < c < 1$.

Then for such c, set $r = \sqrt{c^2 + 1^2}$

Then, $c^2 + 1^2 = r^2 \leftarrow \boxed{\text{So, } P = (0,1) \in {}_cL_r}$

and $0 < c < 1 < 6 \leftarrow \boxed{c < 6}$

and
$$\begin{aligned} c + r &= c + \sqrt{1 + c^2} \\ &< 1 + \sqrt{1 + 1^2} \\ &= 1 + \sqrt{3} \\ &\approx 2.73 \leq 6 \end{aligned}$$

So, if $0 < c < 1$ and $r = \sqrt{c^2 + 1}$,

then $P \in {}_cL_r$ and $\underbrace{{}_cL_r \cap {}_6L = \phi}_{{}_cL_r \text{ is paralell to } {}_6L}$.

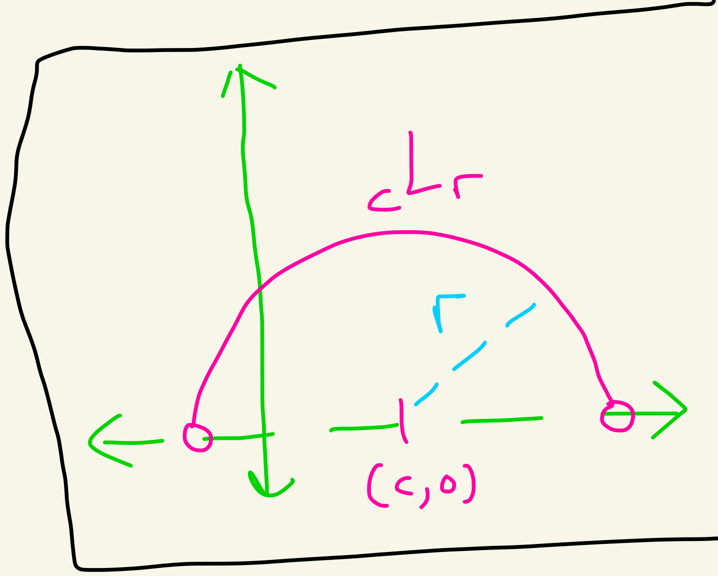
Since there are an infinite number of c with $0 < c < 1$ we get an infinite number of lines that P lies on that are paralell to ${}_6L$.

$\boxed{12(b)}$ Problem 12(a) shows that the following statement is <u>not</u> true by using $P = (0,1)$ and $\ell = {}_6L$.

"Consider the hyperbolic plane $\mathcal{P} = (\mathbb{H}, \mathcal{L}_H)$. Let ℓ be a line in $\mathcal{L}_H$ and $P \in \mathbb{H}$. Then there exists a unique line m where $P \in m$ and m is paralell to ℓ".

13

Let $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ where $x_1 \neq x_2$.



Let $\boxed{c = \dfrac{y_2^2 - y_1^2 + x_2^2 - x_1^2}{2(x_2 - x_1)}}$

and $\boxed{r = \sqrt{(x_1 - c)^2 + y_1^2}}$

We will show that P and Q both lie on $_cL_r$.

Since $r = \sqrt{(x_1-c)^2 + y_1^2}$ we know that $(x_1 - c)^2 + y_1^2 = r^2$ and thus $P = (x_1, y_1)$ lies on $_cL_r$.

Expanding out $(x_1-c)^2 + y_1^2 = r^2$ we get $x_1^2 - 2x_1c + c^2 + y_1^2 = r^2$ which becomes $\boxed{x_1^2 - 2x_1 c + y_1^2 = r^2 - c^2}$ $(*)$

Since $$c = \frac{y_2^2 - y_1^2 + x_2^2 - x_1^2}{2(x_2 - x_1)}$$

we know
$$2x_2c - 2x_1c = y_2^2 - y_1^2 + x_2^2 - x_1^2 .$$

And so,
$$\boxed{x_1^2 - 2x_1c + y_1^2 = x_2^2 - 2x_2c + y_2^2} \quad (**)$$

Now sub $(*)$ into $(**)$ to get
$$r^2 - c^2 = x_2^2 - 2x_2c + y_2^2$$

This gives
$$x_2^2 - 2x_2c + c^2 + y_2^2 = r^2$$

So,
$$(x_2 - c)^2 + y_2^2 = r^2$$

Thus,
$Q = (x_2, y_2)$ also lies on C_r.

(14) We must prove that $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$ is an incidence geometry.

We showed in class that $\mathcal{H}$ is an abstract geometry.

So we must now show properties (i) and (ii) of the incidence geometry definition.

(i) Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be distinct points from $\mathbb{H}$.

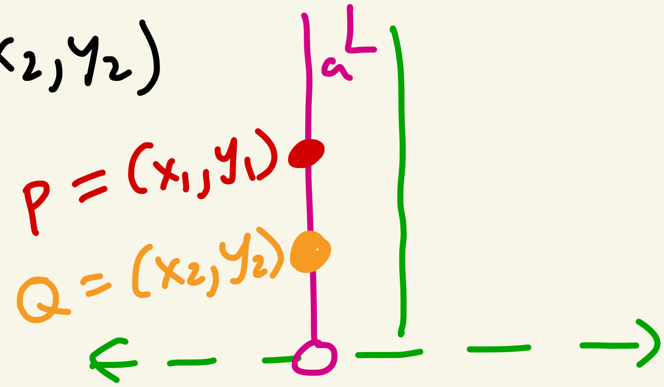
We know from class that there exists a line ℓ where P and Q lie on ℓ.

We must show that ℓ is unique.

Suppose there exist two lines ℓ and m that P and Q both lie on.

We will show that in all cases we have $\ell = m$ and this will show there must be a unique line through P and Q.

Case 1: Suppose ℓ and m are both vertical lines. That is suppose $\ell = {}_aL$ and $m = {}_bL$.



Since $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ both lie on ${}_aL$ we know that $x_1 = a = x_2$.

Since P and Q both lie on ${}_bL$ we know that $x_1 = b = x_2$.

Thus, $a = b$.

So, $\ell = {}_aL = {}_bL = m$.

Case 2: Suppose ℓ is a vertical line and m is a non-vertical line.

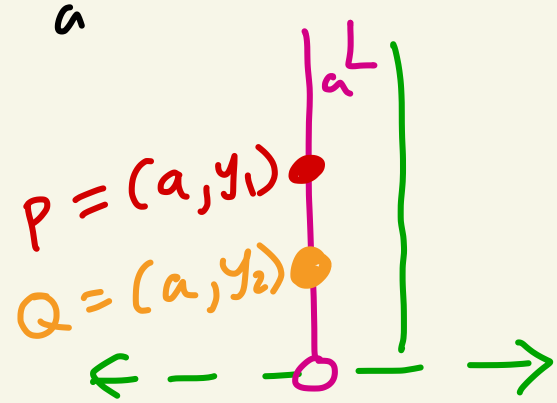
Then, $\ell = {}_aL$ and $m = {}_cL_r$.

Remember, we are assuming that

$P = (x_1, y_1)$ and $Q = (x_2, y_2)$

both lie on ℓ and on m.

Since P, Q both lie on $\ell = {}_aL$ we know



$$P = (x_1, y_1) = (a, y_1)$$

and $Q = (x_2, y_2) = (a, y_2)$

Since P and Q both lie on $m = {}_cL_r$

$(x-c)^2 + y^2 = r^2$
$y > 0$

we know that

$$(a-c)^2 + y_1^2 = r^2$$

$P \in {}_cL_r$

and $(a-c)^2 + y_2^2 = r^2$

$Q \in {}_cL_r$

Subtracting gives $y_1^2 - y_2^2 = 0$.

So, $(y_1 + y_2)(y_1 - y_2) = 0$

Thus either $y_1 + y_2 = 0$ or $y_1 - y_2 = 0$

Suppose $y_1 + y_2 = 0$. $y_2 > 0$ since $Q \in \mathbb{H}$

Then, $y_1 = -y_2 < 0$.

But $y_1 > 0$. since $P \in \mathbb{H}$

We can't have both $y_1 < 0$ and $y_1 > 0$.

Thus, we can't have $y_1 + y_2 = 0$

Suppose $y_1 - y_2 = 0$

Then $y_1 = y_2$.

But then $P = (a, y_1) = (a, y_2) = Q$.

But P and Q were distinct.

Thus, we can't have $y_1 - y_2 = 0$.

Both $y_1 + y_2 = 0$ and $y_1 - y_2 = 0$ can't happen

So we now know that case 2 where ℓ is vertical and m is non-vertical can't happen and we are done with this case.

Case 3: $l = {}_{c_1}L_{r_1}$ and $m = {}_{c_2}L_{r_2}$ are both non-vertical.

Since P and Q both lie on ${}_{c_1}L_{r_1}$ we get:

$$(x_1 - c_1)^2 + y_1^2 = r_1^2 \quad (1)$$
$$(x_2 - c_1)^2 + y_2^2 = r_1^2 \quad (2)$$

If $x_1 = x_2$ then this leads to $y_1^2 - y_2^2 = 0$ by subtracting and using the same method as case 2 we would then get $y_1 = y_2$ and P and Q would not be distinct.

So we can assume $x_1 \neq x_2$.

The above becomes

$$x_1^2 - 2c_1x_1 + c_1^2 + y_1^2 = r_1^2 \quad (1)$$
$$x_2^2 - 2c_1x_2 + c_1^2 + y_2^2 = r_1^2 \quad (2)$$

Then 1 - 2 gives

$$x_1^2 - x_2^2 - 2c_1(x_1 - x_2) + y_1^2 - y_2^2 = 0$$

Then,

$$c_1 = \frac{-y_1^2 + y_2^2 - x_1^2 + x_2^2}{-2(x_1 - x_2)}$$

And

$$r_1 = \sqrt{(x_1 - c_1)^2 + y_1^2}\,.$$

Since P and Q both lie on ${}_{c_2}L_{r_2}$ we can do the same thing as above to

$$\boxed{\begin{aligned}(x_1 - c_1)^2 + y_1^2 &= r_1^2 \\ (x_2 - c_1)^2 + y_2^2 &= r_1^2\end{aligned}}$$

to get that

$$c_2 = \frac{-y_1^2 + y_2^2 - x_1^2 + x_2^2}{-2(x_1 - x_2)}$$

same as c_1

and

$$r_2 = \sqrt{(x_1 - c_2)^2 + y_1^2}\,.$$

this is the same as r_1 since $c_1 = c_2$

Thus, $\ell = {}_{c_1}L_{r_1} = {}_{c_2}L_{r_2} = m$.

$\blacksquare$