Math 3450 - Test 1

Name: Solutions

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T	

1. [3 points] List 3 elements from the set $T = \{x^2 + x - 1 \mid x \in \mathbb{Z}\}$.

$1^2 + 1 - 1 = 1$ $\quad (-1)^2 + (-1) - 1 = -1$

$0^2 + 0 - 1 = -1$ $\quad (-2)^2 + (-2) - 1 = 1$

$2^2 + 2 - 1 = 5$ $\quad 3^2 + 3 - 1 = 11$

2. [15 points - 3 each] Let $A = \{-1, 0, 5, 1, 10, 3\}$, $B = \{1, 10, 3, \pi, \frac{1}{2}\}$, $C = \{0, 5, \pi\}$ and $D = \{1, 5\}$. Compute the following.

(a) $A \cup B = \{-1, 0, 5, 1, 10, 3, \pi, \frac{1}{2}\}$

(b) $B - C. = \{1, 10, 3, \frac{1}{2}\}$

(c) $(A \cap C) \cup (A - C) = \{0, 5\} \cup \{-1, 1, 10, 3\} = \{0, 5, -1, 1, 10, 3\} = A$

(d) $C \times D = \{(0,1), (0,5), (5,1), (5,5), (\pi, 1), (\pi, 5)\}$

(e) The power set $\mathcal{P}(C)$.

$\mathcal{P}(C) = \{\phi, \{0\}, \{5\}, \{\pi\}, \{0,5\}, \{0,\pi\}, \{5,\pi\}, \{0,5,\pi\}\}$

3. [9 points - 3 each]

(a) True or False? Explain why.

$$27 \equiv -33 (\text{mod } 5)$$

True

$27-(-33) = 60$

$5 \mid 60$

So, $27 \equiv -33 \pmod 5$ is True.

(b) True or False? Explain why.

$$\overline{-6} = \overline{27} \quad \text{in } \mathbb{Z}_8$$

False

$-6-27 = -33$

$8 \nmid (-33)$

Thus, $\overline{-6} \neq \overline{27}$.

(c) In $\mathbb{Z}_5 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$, calculate

$$\overline{a} = \overline{2} \cdot \overline{3} + \overline{4} \cdot \overline{4}$$

Reduce $\overline{a}$ so that a satisfies $0 \leq a \leq 4$.

$\overline{a} = \overline{2} \cdot \overline{3} + \overline{4} \cdot \overline{4} = \overline{6} + \overline{16} = \overline{1} + \overline{1} = \overline{2}$

4. [10 points] Let $A_n = \{-2n, 0, 2n\}$.

(a) List the elements in each of the sets A_1, A_2 and A_3.

$A_1 = \{-2, 0, 2\}$

$A_2 = \{-4, 0, 4\}$

$A_3 = \{-6, 0, 6\}$

(b) Calculate $\bigcap_{n=1}^{\infty} A_n$ and $\bigcup_{n=1}^{\infty} A_n$.

$$\bigcap_{n=1}^{\infty} A_n = \{0\}$$

$$\bigcup_{n=1}^{\infty} A_n = \{\ldots, -8, -6, -4, -2, 0, 2, 4, 6, 8, \ldots\}$$

all even integers

5. [10 points]

Pick ONE of the following to prove. Only pick one. If you do both then I will grade (A).

A) Let X and Y be sets. Prove that $\mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y)$.

B) Let $S = \mathbb{N} \times \mathbb{N}$. Define the relation $\sim$ on S where $(a, b) \sim (c, d)$ if and only if $a + d = b + c$. Prove that $\sim$ is an equivalence relation on S.

See HW solutions

6. [10 points] Let A, B, and C be sets. Prove that

$$A \times (B - C) \subseteq (A \times B) - (A \times C).$$

Let $(x,y) \in A \times (B-C)$.

Then $x \in A$ and $y \in B - C$.

Thus, $x \in A$ and $y \in B$ and $y \notin C$.

Thus, $(x,y) \in A \times B$ and $(x,y) \notin A \times C$.

So, $(x,y) \in (A \times B) - (A \times C)$.

7. [10 points] Consider the set of integers $\mathbb{Z}$. Let $n \in \mathbb{Z}$ with $n \geq 2$. Given $a, b \in \mathbb{Z}$, define $a \sim b$ if and only if n divides $a - b$. Prove that $\sim$ is an equivalence relation.

reflexive

Let $a \in \mathbb{Z}$.
Then, $a - a = 0 = n \cdot 0$.
So, $n \mid (a-a)$.
Thus, $a \sim a$.

symmetric

Let $a, b \in \mathbb{Z}$ and suppose $a \sim b$.
Then $n \mid (a-b)$.
So, $a - b = nk$ for some $k \in \mathbb{Z}$.
Thus, $b - a = n(-k)$. Note $-k \in \mathbb{Z}$.
So, $n \mid (b-a)$.
Thus, $b \sim a$.

transitive

Let $a, b, c \in \mathbb{Z}$ and suppose $a \sim b$ and $b \sim c$.
Then $n \mid (a-b)$ and $n \mid (b-c)$.
Thus, there exist $k, l \in \mathbb{Z}$ with $nk = a-b$ and $nl = b-c$.
Adding these equations gives $nk + nl = a - b + b - c$.
So, $n(k+l) = a - c$. Note that $k + l \in \mathbb{Z}$.
Thus, $n \mid (a-c)$.
So, $a \sim c$.