Math 3450 - Homework # 3
Equivalence Relations and Well-Defined Operations

1. A set S and a relation $\sim$ on S is given. For each example, check if $\sim$ is (i) reflexive, (ii) symmetric, and/or (iii) transitive. If $\sim$ satisfies the property that you are checking, then prove it. If $\sim$ does not satisfy the property that you are checking, then give an example to show it.

   (a) $S = \mathbb{R}$ where $a \sim b$ if and only if $a \leq b$.

   Solution:

   (i) Yes, $\sim$ is reflexive. Proof: Let $a \in \mathbb{R}$. Then $a \leq a$. So $a \sim a$.

   (ii) No, $\sim$ is not symmetric. Counterexample: $3 \leq 4$, but $4 \not\leq 3$. That is, $3 \sim 4$ but $4 \not\sim 3$.

   (iii) Yes, $\sim$ is transitive. Proof: Let $a, b, c \in \mathbb{R}$ and suppose that $a \sim b$ and $b \sim c$. Then $a \leq b$ and $b \leq c$. So $a \leq c$. Thus $a \sim c$.

   (b) $S = \mathbb{R}$ where $a \sim b$ if and only if $|a| = |b|$.

   Solution:

   (i) Yes, $\sim$ is reflexive. Proof: Let $a \in \mathbb{R}$. Then $|a| = |a|$. So $a \sim a$.

   (ii) Yes, $\sim$ is symmetric. Proof: Let $a, b \in \mathbb{R}$ and suppose that $a \sim b$. Then $|a| = |b|$. So $|b| = |a|$. Thus $b \sim a$.

   (iii) Yes, $\sim$ is transitive. Proof: Let $a, b, c \in \mathbb{R}$ and suppose that $a \sim b$ and $b \sim c$. Then $|a| = |b|$ and $|b| = |c|$. So $|a| = |c|$. Thus $a \sim c$.

   (c) $S = \mathbb{Z}$ where $a \sim b$ if and only if $a|b$.

   Solution:

   (i) Yes, $\sim$ is reflexive. Proof: Let $a \in \mathbb{Z}$. Then $a(1) = a$. Hence $a|a$. So $a \sim a$.

   (ii) No, $\sim$ is not symmetric. Counterexample: $3|6$, but $6 \nmid 3$.

   (iii) Yes, $\sim$ is transitive. Proof: Let $a, b, c \in \mathbb{Z}$. Suppose that $a \sim b$ and $b \sim c$. Then $a|b$ and $b|c$. Thus there exists $k, m \in \mathbb{Z}$ such that $ak = b$ and $bm = c$. Then $c = bm = (ak)m = a(km)$. So $a|c$. Thus $a \sim c$.

   (d) S is the set of subsets of $\mathbb{N}$ where $A \sim B$ if and only if $A \subseteq B$. Some examples of elements of S are $\{1, 10, 199\}$, $\{2, 7, 10\}$, and $\{2, 10, 3, 7\}$. Note that $\{2, 7, 10\} \sim \{2, 10, 3, 7\}$

Solution:

(i) Yes, $\sim$ is reflexive. Proof: $A \subseteq A$ for all $A \in S$.

(ii) No, $\sim$ is not symmetric. Counterexample: $\{3\} \subseteq \{3, 42\}$, but $\{3, 42\} \not\subseteq \{3\}$.

(iii) Yes, $\sim$ is transitive. Proof: Let $A, B, C \in S$ with $A \sim B$ and $B \sim C$. Then $A \subseteq B$ and $B \subseteq C$. We want to show that $A \subseteq C$. Let $x \in A$. Since $A \subseteq B$, we have that $x \in B$. Since $B \subseteq C$ we have that $x \in C$. So $A \subseteq C$ and thus $A \sim C$.

2. Consider the set $S = \mathbb{R}$ where $x \sim y$ if and only if $x^2 = y^2$.

   (a) Find all the numbers that are related to $x = 1$. Repeat this exercise for $x = \sqrt{2}$ and $x = 0$.

   Solution:

   $1 \sim 1$ since $1^2 = 1^2$. We also have $1 \sim (-1)$ since $1^2 = (-1)^2$. There are no other elements related to 1.

   $\sqrt{2} \sim \sqrt{2}$ since $(\sqrt{2})^2 = (\sqrt{2})^2$. We also have $\sqrt{2} \sim (-\sqrt{2})$ since $(\sqrt{2})^2 = (-\sqrt{2})^2$. There are no other elements related to $\sqrt{2}$.

   $0 \sim 0$ since $0^2 = 0^2$. There are no other elements related to 0.

   (b) Prove that $\sim$ is an equivalence relation on S.

   Solution:

   Proof. Reflexive: We know that $x^2 = x^2$ for all real numbers x. Therefore $x \sim x$ for all real numbers x. So $\sim$ is reflexive.

   Symmetric: Let $x, y \in \mathbb{R}$. Suppose that $x \sim y$.
   Since $x \sim y$ we have that $x^2 = y^2$.
   So $y^2 = x^2$.
   Therefore $y \sim x$.

   Transitive Let $x, y, z \in \mathbb{R}$. Suppose that $x \sim y$ and $y \sim z$.
   Since $x \sim y$ we have that $x^2 = y^2$.
   Since $y \sim z$ we have that $y^2 = z^2$.
   So $x^2 = y^2 = z^2$.
   Therefore $x \sim z$. □

   (c) Draw a number line. Draw a picture of the equivalence class of 1. Repeat this for $x = 0$, $x = \sqrt{6}$, $x = -3$.

   Solution: Please draw a picture.

(d) Describe the elements of $S/\sim$.

Solution:

If $x \neq 0$, then the equivalence class of x is $\overline{x} = \{-x, x\}$. The equivalence class of 0 is $\overline{0} = \{0\}$.

3. Consider the set $S = \mathbb{Z}$ where $x \sim y$ if and only if $2|(x+y)$.

(a) List six numbers that are related to $x = 2$.

Solution:

$2 \sim (-4)$ since $2|(2+(-4))$.
$2 \sim (-2)$ since $2|(2+(-2))$.
$2 \sim (0)$ since $2|(2+(0))$.
$2 \sim (2)$ since $2|(2+(2))$.
$2 \sim (4)$ since $2|(2+(4))$.
$2 \sim (6)$ since $2|(2+(6))$.

(b) Prove that $\sim$ is an equivalence relation on S.

Proof. Reflexive: Let $x \in \mathbb{Z}$.
Since $2|2x$ we have that $2|(x+x)$.
So $x \sim x$.
Symmetric: Let $x, y \in \mathbb{Z}$ and suppose that $x \sim y$.
Thus $2|(x+y)$.
So $2|(y+x)$.
So $y \sim x$.
Transitive: Let $x, y, z \in \mathbb{Z}$ and suppose that $x \sim y$ and $y \sim z$.
Therefore $2|(x+y)$ and $2|(y+z)$.
So there exist $k, \ell \in \mathbb{Z}$ such that $2k = x+y$ and $2\ell = y+z$.
Add these equations to get $2k + 2\ell = x + 2y + z$.
Subtract $2y$ from both sides to get $2(k+\ell-y) = x+z$.
Note that $k + \ell - y \in \mathbb{Z}$, because $k, \ell, y \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under addition and subtraction.
So $2|(x+z)$.
So $x \sim z$.

$\square$

(c) Draw a picture of the set of integers. Next, circle the numbers that are in the equivalence class of -3.

Solution: Draw a picture and circle these numbers:

$\ldots, -7, -5, -3, -1, 1, 3, 5, 7, \ldots$

(d) Describe the elements of $S/\sim$. Draw a picture of several equivalence classes.

Solution: Draw a picture of the following:

$$\begin{aligned} \overline{0} &= \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\} = \overline{-2} = \overline{2} = \overline{4} = \overline{-4} = \cdots \\ \overline{1} &= \{\ldots, -7, -5, -3, -1, 1, 3, 5, 7, \ldots\} = \overline{-1} = \overline{3} = \overline{-3} = \overline{-5} = \cdots \end{aligned}$$

So $S/\sim$ is equal to $\{\overline{0}, \overline{1}\}$. That is, one equivalence class is the set of all odd numbers; the other equivalence class is the set of all even numbers.

4. Show that the operation $\overline{a} \oplus \overline{b} = \overline{a}^2 + \overline{b}^2$ is a well-defined operation for $\mathbb{Z}_n$. Here $\overline{a}^2$ means $\overline{a} \cdot \overline{a}$. For example, in $\mathbb{Z}_4$ we have that

$$\overline{2} \oplus \overline{3} = \overline{2} \cdot \overline{2} + \overline{3} \cdot \overline{3} = \overline{4} + \overline{9} = \overline{1}.$$

Proof. 1) Let $\overline{a}, \overline{b} \in \mathbb{Z}_n$ where $a, b \in \mathbb{Z}$.

Then

$$\overline{a} \oplus \overline{b} = \overline{a}^2 + \overline{b}^2 = \overline{a^2} + \overline{b^2} = \overline{a^2 + b^2}.$$

Since $a, b \in \mathbb{Z}$ we have that $a^2 + b^2 \in \mathbb{Z}$.

Therefore, $\overline{a} \oplus \overline{b} = \overline{a^2 + b^2} \in \mathbb{Z}_n$.

So $\mathbb{Z}_n$ is closed under the operation $\oplus$.

2) Suppose that $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ such that $\overline{a_1} = \overline{a_2}$ and $\overline{b_1} = \overline{b_2}$. We need to show that $\overline{a_1} \oplus \overline{b_1} = \overline{a_2} \oplus \overline{b_2}$.

From class we had a theorem that says that if $\overline{x} = \overline{y}$ and $\overline{w} = \overline{z}$, then $\overline{x} + \overline{w} = \overline{y} + \overline{z}$ and $\overline{x} \cdot \overline{w} = \overline{y} \cdot \overline{z}$.

Repeatedly using the above theorem we get the following.

We have that $\overline{a_1} \cdot \overline{a_1} = \overline{a_2} \cdot \overline{a_2}$ by multiplying the equations $\overline{a_1} = \overline{a_2}$ and $\overline{a_1} = \overline{a_2}$.

Similarly, $\overline{b_1} \cdot \overline{b_1} = \overline{b_2} \cdot \overline{b_2}$ by multiplying the equations $\overline{b_1} = \overline{b_2}$ and $\overline{b_1} = \overline{b_2}$.

Adding the two equations above we get that $\overline{a_1}\cdot\overline{a_1}+\overline{b_1}\cdot\overline{b_1} = \overline{a_2}\cdot\overline{a_2}+\overline{b_2}\cdot\overline{b_2}$.

Therefore, $\overline{a_1} \oplus \overline{b_1} = \overline{a_2} \oplus \overline{b_2}$.

Thus $\oplus$ is a well-defined operation on $\mathbb{Z}_n$. □

5. Given two integers a and b, let $\min(a, b)$ denote the minimum (smaller) of a and b. Let n be an integer with $n \geq 2$. Is the operation $\overline{a} \oplus \overline{b} = \overline{\min(a, b)}$ a well-defined operation on $\mathbb{Z}_n$?

   Solution: This operation is not well-defined. For example, consider $n = 4$. In $\mathbb{Z}_4$ we have that $\overline{0} = \overline{8}$ and $\overline{1} = \overline{5}$. Thus, for the operation to be well-defined we would need $\overline{0} \oplus \overline{1} = \overline{8} \oplus \overline{5}$. However, $\overline{0} \oplus \overline{1} = \overline{\min(0, 1)} = \overline{0}$ and $\overline{8} \oplus \overline{5} = \overline{\min(8, 5)} = \overline{5}$. But $\overline{0} \neq \overline{5}$ in $\mathbb{Z}_4$.

6. (a) Show that the operation $\frac{a}{b} \oplus \frac{c}{d} = \frac{ad}{bc}$ is not a well-defined operation on $\mathbb{Q}$. (b) Is the operation well-defined on $\mathbb{Q} - \{0\}$?

   (a) Show that the operation $\frac{a}{b} \oplus \frac{c}{d} = \frac{ad}{bc}$ is not a well-defined operation on $\mathbb{Q}$.
   Solution: We have that $\frac{5}{2}, \frac{0}{1} \in \mathbb{Q}$ however $\frac{5}{2} \oplus \frac{0}{1} = \frac{5\cdot 1}{2\cdot 0} = \frac{5}{0} \notin \mathbb{Q}$. Hence $\mathbb{Q}$ is not closed under $\oplus$ and the operation is not well-defined.

   (b) Is the operation well-defined on $\mathbb{Q} \setminus \{0\}$?
   Solution: Yes! Here is a proof.

   Proof. 1) Let $a, b, c, d \in \mathbb{Z}$ with $a \neq 0$, $b \neq 0$,$c \neq 0$,$d \neq 0$ so that $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q} - \{0\}$.
   Since $a \neq 0$, $b \neq 0$,$c \neq 0$,$d \neq 0$ we have that $ad \neq 0$ and $bc \neq 0$.
   Thus $\frac{a}{b} \oplus \frac{c}{d} = \frac{ad}{bc} \in \mathbb{Q} - \{0\}$.
   Therefore $\mathbb{Q} - \{0\}$ is closed under the operation $\oplus$.
   2) Suppose further that we have $e, f, g, h \in \mathbb{Z}$ with $e \neq 0$, $f \neq 0$,$g \neq 0$,$h \neq 0$ so that $\frac{e}{f}, \frac{g}{h} \in \mathbb{Q} - \{0\}$.
   Also assume that $\frac{a}{b} = \frac{e}{f}$ and $\frac{c}{d} = \frac{g}{h}$.
   We want to show that $\frac{a}{b} \oplus \frac{c}{d} = \frac{e}{f} \oplus \frac{g}{h}$.
   We have that $\frac{a}{b} \oplus \frac{c}{d} = \frac{ad}{bc}$ and $\frac{e}{f} \oplus \frac{g}{h} = \frac{eh}{fg}$.

Since $\frac{a}{b} = \frac{e}{f}$ we have that $af = be$.
Since $\frac{c}{d} = \frac{g}{h}$ we have that $ch = dg$.
Multiplying $af = be$ by $dg = ch$ we get $afdg = bech$.
Rearranging we get $(ad)(fg) = (bc)(eh)$.
Therefore, $\frac{ad}{bc} = \frac{eh}{fg}$.
So $\frac{a}{b} \oplus \frac{c}{d} = \frac{e}{f} \oplus \frac{g}{h}$.
Thus, the operation is well-defined.

□

7. Is the operation $\overline{a} \oplus \overline{b} = \overline{a^b}$ a well-defined operation on $\mathbb{Z}_n$?

   Solution: There are two issues with this operation.

   One issue is as follows. As an example, consider $n = 4$. In $\mathbb{Z}_4$ we have that $\overline{1} = \overline{5}$. Thus, for the operation to be well-defined we must have that $\overline{2} \oplus \overline{1} = \overline{2} \oplus \overline{5}$. However, $\overline{2} \oplus \overline{1} = \overline{2^1} = \overline{2}$ and $\overline{2} \oplus \overline{5} = \overline{2^5} = \overline{32} = \overline{0}$. And $\overline{2} \neq \overline{0}$ in $\mathbb{Z}_4$.

   Another issue is when b is a negative integer. For example, in $\mathbb{Z}_4$ suppose we want to calculate $\overline{2} \oplus \overline{-1}$. What does this mean? The formula says that it is $\overline{2^{-1}}$. But what is that in $\mathbb{Z}_4$? In fact there is no way to make sense of $1/2$ in $\mathbb{Z}_4$ because there is no multiplicative inverse for $\overline{2}$ in $\mathbb{Z}_4$. (Why?) Because there is no $\overline{x} \in \mathbb{Z}_4$ with $\overline{x} \cdot \overline{2} = \overline{1}$. We can check:

   $\overline{0} \cdot \overline{2} = \overline{0} \neq \overline{1}$

   $\overline{1} \cdot \overline{2} = \overline{2} \neq \overline{1}$

   $\overline{2} \cdot \overline{2} = \overline{4} = \overline{0} \neq \overline{1}$

   $\overline{3} \cdot \overline{2} = \overline{6} = \overline{2} \neq \overline{1}$

   Thus there is no way to define $\overline{2^{-1}}$ in $\mathbb{Z}_4$.

8. (Constructing the integers from the natural numbers) Let $S = \mathbb{N} \times \mathbb{N}$. Define the relation $\sim$ on S where $(a, b) \sim (c, d)$ if and only if $a+d = b+c$.

   (a) Is $(3, 6) \sim (7, 10)$?
   Solution: Yes, because $3 + 10 = 6 + 7$.

   (b) Is $(1, 1) \sim (3, 5)$?
   Solution: No, because $1 + 5 \neq 1 + 3$.

(c) Prove that $\sim$ is an equivalence relation.

Proof. Reflexive: Let $(a, b) \in \mathbb{N} \times \mathbb{N}$.
Then $a + b = b + a$.
So $(a, b) \sim (a, b)$.
Symmetric: Let $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$.
Suppose $(a, b) \sim (c, d)$.
We know that $a + d = b + c$, because $(a, b) \sim (c, d)$.
So $c + b = d + a$.
So $(c, d) \sim (a, b)$.
Transitive: Let $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$.
Suppose that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.
We know that $a+d = b+c$ and $c+f = d+e$, because $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.
Add these two equations to get $a + c + d + f = b + c + d + e$.
Subtract $c + d$ from both sides to get $a + f = b + e$.
So $(a, b) \sim (e, f)$.

Therefore, $\sim$ is an equivalence relation, because it is reflexive, symmetric, and transitive.

□

(d) List five elements from each of the following equivalence classes: $\overline{(1, 1)}$, $\overline{(1, 2)}$, $\overline{(5, 12)}$.

Solution: Some possible answers:
$(2, 2), (3, 3), (4, 4), (5, 5), (47, 47) \in \overline{(1, 1)}$.
$(2, 3), (3, 4), (4, 5), (5, 6), (47, 48) \in \overline{(1, 2)}$.
$(2, 9), (3, 10), (4, 11), (5, 12), (47, 56) \in \overline{(5, 12)}$.

(e) Define the operation $\overline{(a, b)} \oplus \overline{(c, d)} = \overline{(a + c, b + d)}$. Prove that $\oplus$ is well-defined on the set of equivalence classes.

Proof. 1) Consider two equivalence classes $\overline{(a, b)}$ and $\overline{(c, d)}$ where $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$.
Then $a + c$ and $b + d$ are both in $\mathbb{N}$ because $\mathbb{N}$ is closed under addition.

Thus, $\overline{(a,b)} \oplus \overline{(c,d)} = \overline{(a+c,b+d)}$ is a valid equivalence class in $\mathbb{N} \times \mathbb{N}/\sim$.

2) Now suppose that $\overline{(a,b)}$,$\overline{(c,d)}$,$\overline{(e,f)}$,and $\overline{(g,h)}$ are equivalence classes of $\mathbb{N} \times \mathbb{N}/\sim$.
Further suppose that $\overline{(a,b)} = \overline{(e,f)}$ and $\overline{(c,d)} = \overline{(g,h)}$.
We need to show that $\overline{(a,b)} \oplus \overline{(c,d)} = \overline{(e,f)} \oplus \overline{(g,h)}$.
We have that $a+f = b+e$ since $\overline{(a,b)} = \overline{(e,f)}$.
We also have that $c+h = d+g$ since $\overline{(c,d)} = \overline{(g,h)}$.
Adding these two equations gives $a+f+c+h = b+e+d+g$.
Rearranging gives $(a+c)+(f+h) = (b+d)+(e+g)$.
Therefore, $\overline{(a+c,b+d)} = \overline{(e+g,f+h)}$.
Hence $\overline{(a,b)} \oplus \overline{(c,d)} = \overline{(e,f)} \oplus \overline{(g,h)}$.

The above arguments show that $\oplus$ is a well-defined operation on the equivalence classes of $\mathbb{N} \times \mathbb{N}/\sim$.

□

9. (Constructing the rational numbers from the integers) Let $S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Define the relation $\sim$ on S where $(a,b) \sim (c,d)$ if and only if $ad = bc$.

   (a) Is $(1,5) \sim (-3,-15)$?
   Solution: Yes, because $1(-15) = 5(-3)$.

   (b) Is $(-1,1) \sim (2,3)$?
   Solution: No, because $(-1)(3) \neq 1(2)$.

   (c) Prove that $\sim$ is an equivalence relation.

   Proof. Reflexive: Let $(a,b) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$.
   Then $ab = ba$.
   So $(a,b) \sim (a,b)$.
   Symmetric: Let $(a,b),(c,d) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$.
   Suppose $(a,b) \sim (c,d)$.
   We know that $ad = bc$, because $(a,b) \sim (c,d)$.
   So $cb = da$.

Hence $(c, d) \sim (a, b)$.
Transitive: Let $(a, b), (c, d), (e, f) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$.
Note that $d \neq 0$ and $f \neq 0$ since $d, f \in \mathbb{Z} - \{0\}$.
Suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.
We know that $ad = bc$ and $cf = de$, because $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.
Thus

$$ad = bc = b\left(\frac{de}{f}\right) = \frac{bde}{f}$$

Thus $adf = bde$.
Since $d \neq 0$ we can divide by d to get $af = be$.
So $(a, b) \sim (e, f)$ since $af = be$.

Therefore, $\sim$ is an equivalence relation, because it is reflexive, symmetric, and transitive.

□

(d) List five elements from each of the following equivalence classes: $\overline{(1,1)}$, $\overline{(0,2)}$, $\overline{(2,3)}$.

Solution: Some possible answers:
$(2, 2), (3, 3), (4, 4), (5, 5), (47, 47) \in \overline{(1, 1)}$.
$(0, 1), (0, 2), (0, -1), (0, -2), (0, -47) \in \overline{(0, 2)}$.
$(2, 3), (4, 6), (6, 9), (-2, -3), (-4, -6) \in \overline{(2, 3)}$.

(e) Define the operation $\overline{(a, b)} \oplus \overline{(c, d)} = \overline{(ad + bc, bd)}$. Prove that $\oplus$ is well-defined on the set of equivalence classes.

Proof. 1) Consider two equivalence classes $\overline{(a, b)}$ and $\overline{(c, d)}$ where $(a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$.
Then $ad + bc \in \mathbb{Z}$ because $a, b, c, d \in \mathbb{Z}$ and the integers are closed under addition and multiplication.
Also, since $b, d \in \mathbb{Z} - \{0\}$ we have that $bd \neq 0$ and so $bd \in \mathbb{Z} - \{0\}$.
Thus $(ad+bc, bd) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$ and $\overline{(a, b)} \oplus \overline{(c, d)} = \overline{(ad + bc, bd)}$ is a valid equivalence class.

2) Now suppose that $\overline{(a, b)}$,$\overline{(c, d)}$,$\overline{(x, y)}$,and $\overline{(w, z)}$ are equivalence classes in $\mathbb{Z} \times (\mathbb{Z} - \{0\})/\sim$.

Further suppose that $\overline{(a,b)} = \overline{(x,y)}$ and $\overline{(c,d)} = \overline{(w,z)}$.
We need to show that $\overline{(a,b)} \oplus \overline{(c,d)} = \overline{(x,y)} \oplus \overline{(w,z)}$.
That is, we need to show that $\overline{(ad+bc,bd)} = \overline{(xz+yw,yz)}$.
The above is equivalent to showing that $(ad+bc)yz = bd(xz+yw)$.
Let's do this.
Since $\overline{(a,b)} = \overline{(x,y)}$ we have that $ay = bx$.
Since $\overline{(c,d)} = \overline{(w,z)}$ we have that $cz = dw$.
Therefore, using the equations $ay = bx$ and $cz = dw$ we get that

$$\begin{aligned}(ad+bc)yz &= adyz + bcyz \\ &= (ay)(dz) + (cz)(by) \\ &= (bx)(dz) + (dw)(by) \\ &= bd(xz+yw).\end{aligned}$$

Thus, $\overline{(ad+bc,bd)} = \overline{(xz+yw,yz)}$.
Thus, the operation $\oplus$ is well-defined on the equivalence classes of $\mathbb{Z} \times (\mathbb{Z} - \{0\})/\sim$.

□

(f) Define the operation $\overline{(a,b)} \odot \overline{(c,d)} = \overline{(ac,bd)}$. Prove that $\odot$ is well-defined on the set of equivalence classes.

Proof. 1) Consider two equivalence classes $\overline{(a,b)}$ and $\overline{(c,d)}$ where $(a,b),(c,d) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$.
Then $ac \in \mathbb{Z}$ because $a, c \in \mathbb{Z}$ and the integers are closed under multiplication.
Also, since $b, d \in \mathbb{Z} - \{0\}$ we have that $bd \neq 0$ and so $bd \in \mathbb{Z} - \{0\}$.
Thus $(ac, bd) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$ and $\overline{(a,b)} \odot \overline{(c,d)} = \overline{(ac,bd)}$ is a valid equivalence class.

2) Now suppose that $\overline{(a,b)}$,$\overline{(c,d)}$,$\overline{(x,y)}$,and $\overline{(w,z)}$ are equivalence classes in $\mathbb{Z} \times (\mathbb{Z} - \{0\})/\sim$.
Further suppose that $\overline{(a,b)} = \overline{(x,y)}$ and $\overline{(c,d)} = \overline{(w,z)}$.
We need to show that $\overline{(a,b)} \odot \overline{(c,d)} = \overline{(x,y)} \odot \overline{(w,z)}$.
That is, we need to show that $\overline{(ac,bd)} = \overline{(xw,yz)}$.

The above is equivalent to showing that $(ac)(yz) = (bd)(xw)$.
Let's do this.
Since $\overline{(a,b)} = \overline{(x,y)}$ we have that $ay = bx$.
Since $\overline{(c,d)} = \overline{(w,z)}$ we have that $cz = dw$.
Therefore, using the equations $ay = bx$ and $cz = dw$ we get that

$$(ac)(yz) = (ay)(cz) = (bx)(dw) = (bd)(xw).$$

Thus, $\overline{(ac, bd)} = \overline{(xw, yz)}$.
Therefore, the operation $\odot$ is well-defined on the equivalence classes of $\mathbb{Z} \times (\mathbb{Z} - \{0\})/\sim$.

$\square$