Math 3450

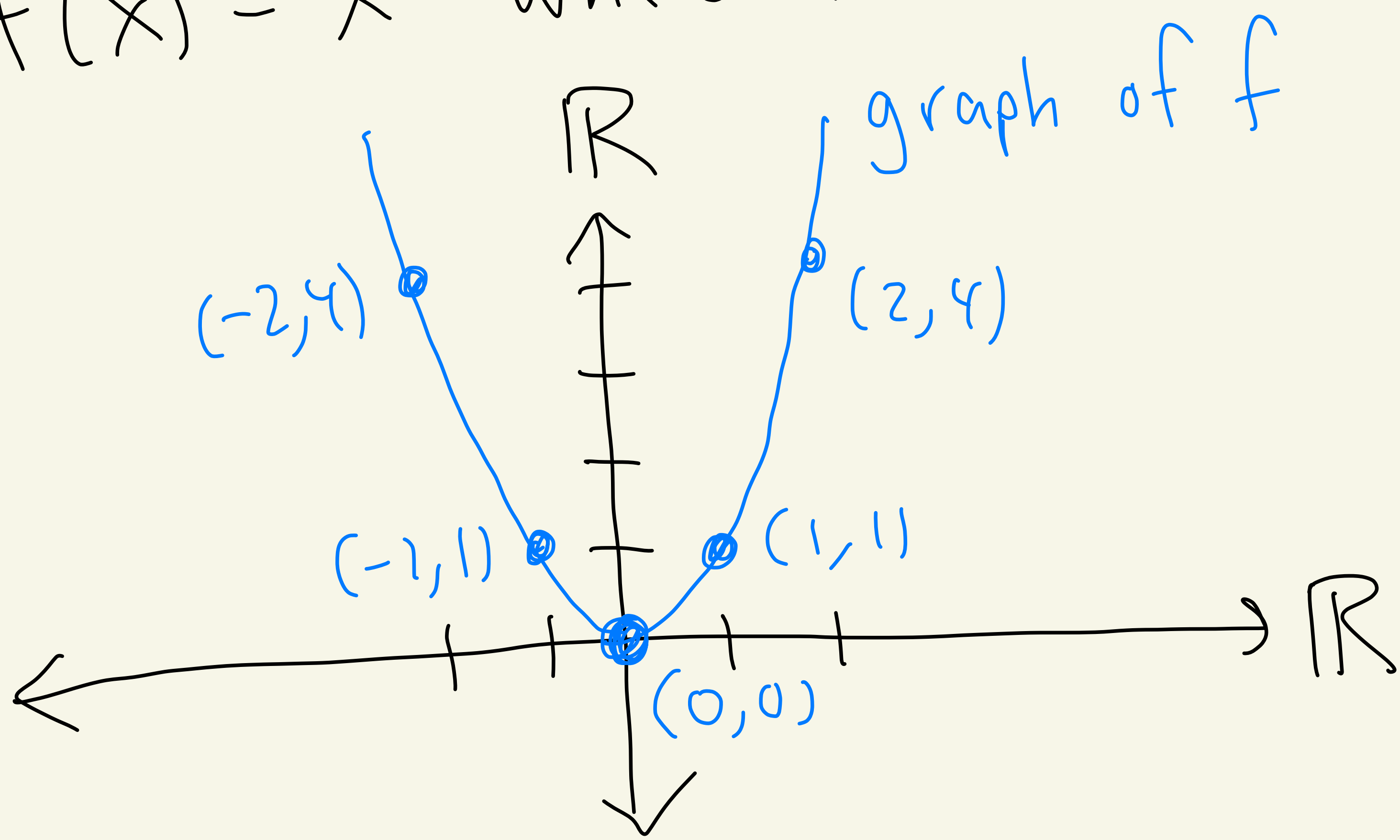
3/5/24

3/5	3/7
3/12 Review Day	3/14 Test 1

Topic 4 – Functions

We are going to formally define functions as sets but then after that we won't really use that method anymore we will just use formulas like usual.

Ex: Consider the function
$f(x) = x^2$ where $x \in \mathbb{R}$.



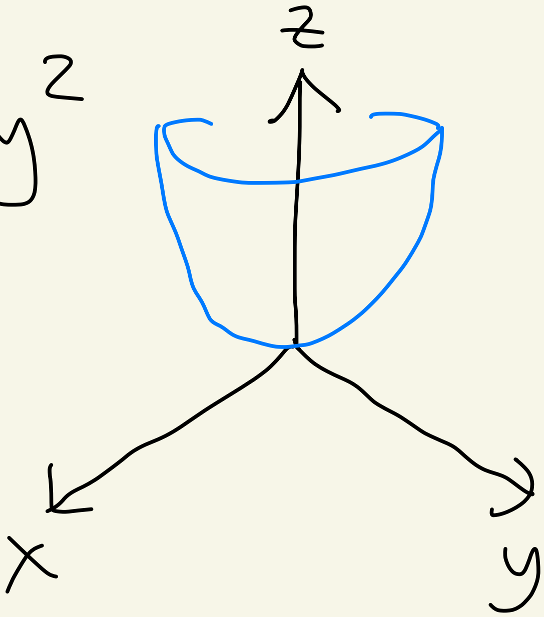
The graph is

$$\{(x, x^2) \mid x \in \mathbb{R}\}$$

This graph lives inside of

$$\underset{\text{domain}}{\mathbb{R}} \times \underset{\text{co-domain, where the range lives}}{\mathbb{R}}$$

Ex: $f(x,y) = x^2 + y^2$



graph lives in

$$\mathbb{R}^3 = \underbrace{\mathbb{R}\times\mathbb{R}}_{\text{domain}} \times \underbrace{\mathbb{R}}_{\text{co-domain}}$$

Def: Let A and B be sets. Let f be a subset of $A \times B$. We say that <u>f is a function from A to B</u> if

(1) for every $a \in A$ there exists $b \in B$ where $(a,b) \in f$

this is saying that we can plug a into f to get b, ie $f(a) = b$

and

(2) if $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$

vertical line test

If this is the case then we write $f: A \to B$ to mean that f is a function from A to B

The set A is called the
<u>domain</u> of f.

The set B is called the
<u>co-domain</u> of f.

If $(a,b) \in f$ then
we write $f(a) = b$

The <u>range</u> of f is

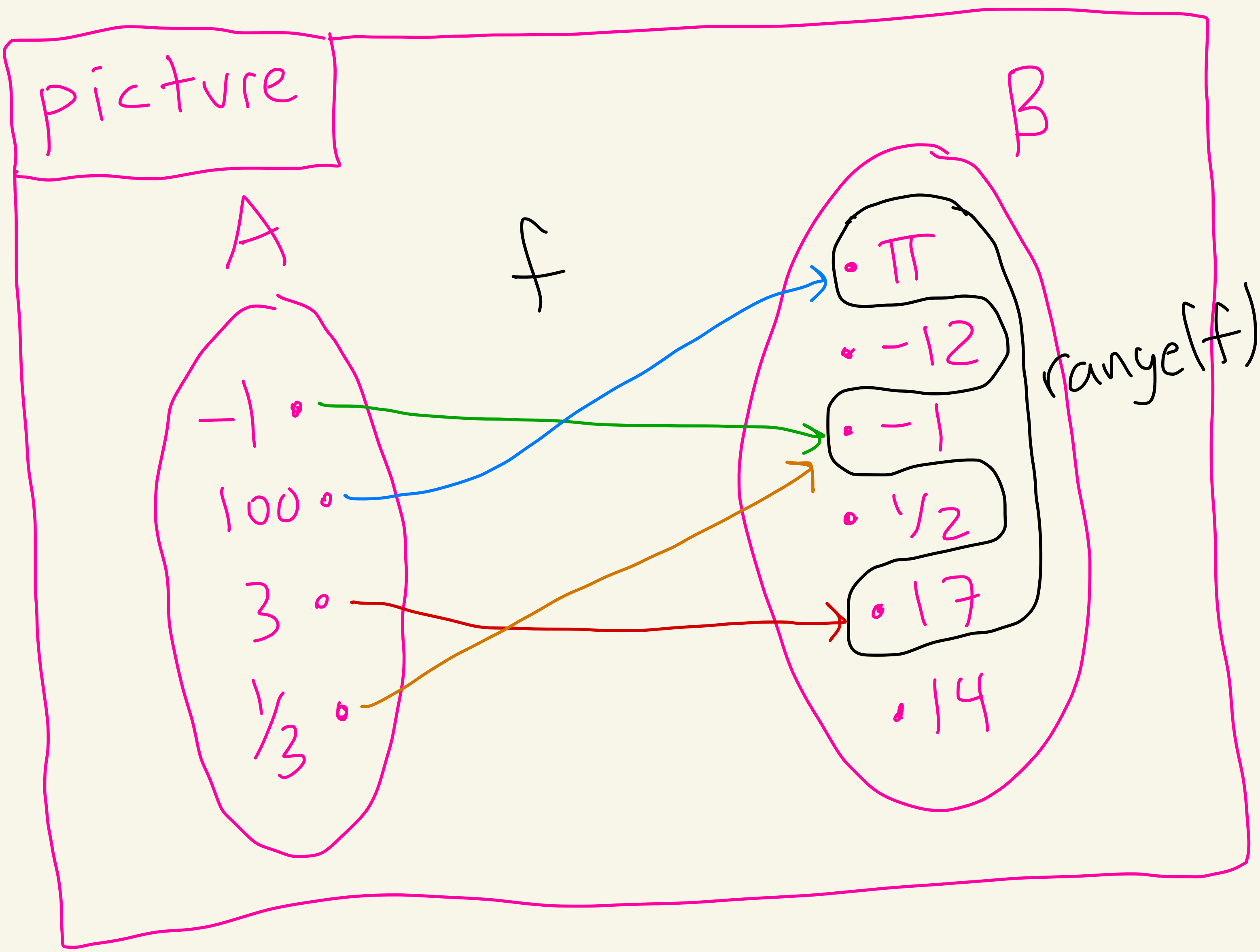
$$\text{range}(f) = \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists } a \in A \\ \text{with } f(a) = b \end{array} \right\}$$

Ex: $A = \{-1, 100, 3, \frac{1}{3}\}$

$B = \{\pi, -12, -1, \frac{1}{2}, 17, 14\}$

$f = \{(-1,-1), (100,\pi), (3,17), (\frac{1}{3},-1)\}$

$f(-1) = -1$ $\quad f(100) = \pi$ $\quad f(3) = 17$ $\quad f(\frac{1}{3}) = -1$



Is f a function from A to B?

1. f is defined on all of A
2. no element of A gets mapped to more than one element of B

Yes, f is a function from A to B.

$$\text{domain}(f) = A$$

$$\text{co-domain}(f) = B$$

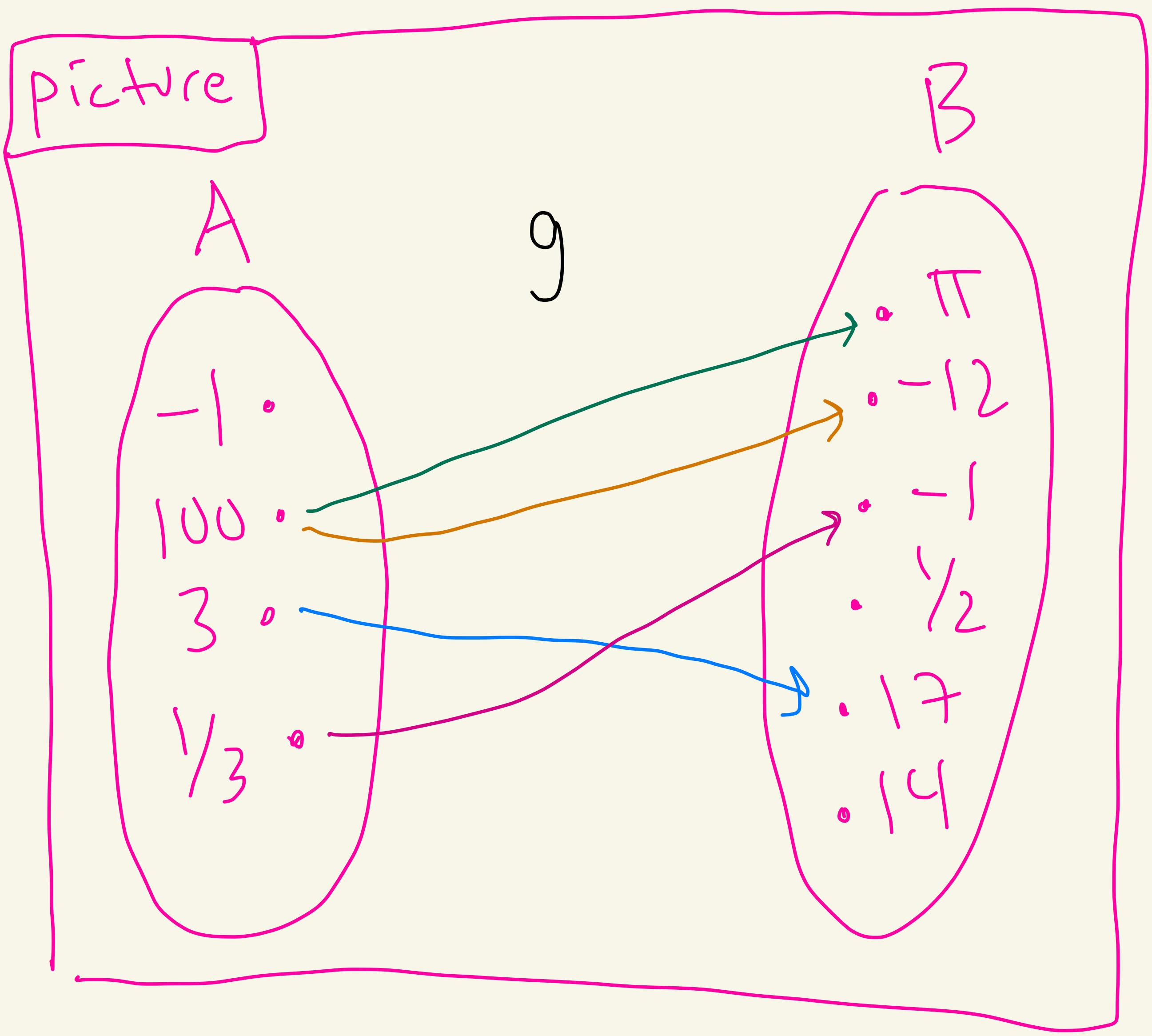
$$\text{range}(f) = \underbrace{\{\pi, -1, 17\}}_{\text{subset of co-domain } B}$$

<u>Ex:</u> $A = \{-1, 100, 3, \frac{1}{3}\}$

$B = \{\pi, -12, -1, \frac{1}{2}, 17, 14\}$

$g = \{(100,\pi), (3,17), (\frac{1}{3},-1), (100,-12)\}$

$g(100)=\pi$ $\quad g(3)=17$ $\quad g(\frac{1}{3})=-1$ $\quad g(100)=-12$



① $g(-1)$ is not defined ✗

(2) $g(100)$ has two values: π & -12 ✗

g is not a function from A to B.

Let's now use formulas to define functions instead of defining them as subsets of $A \times B$.

Ex: Let A be any non-empty set. The identity function on A is the function

$$i_A : A \to A$$

defined as

$$i_A(x) = x \quad \text{for all } x \in A.$$

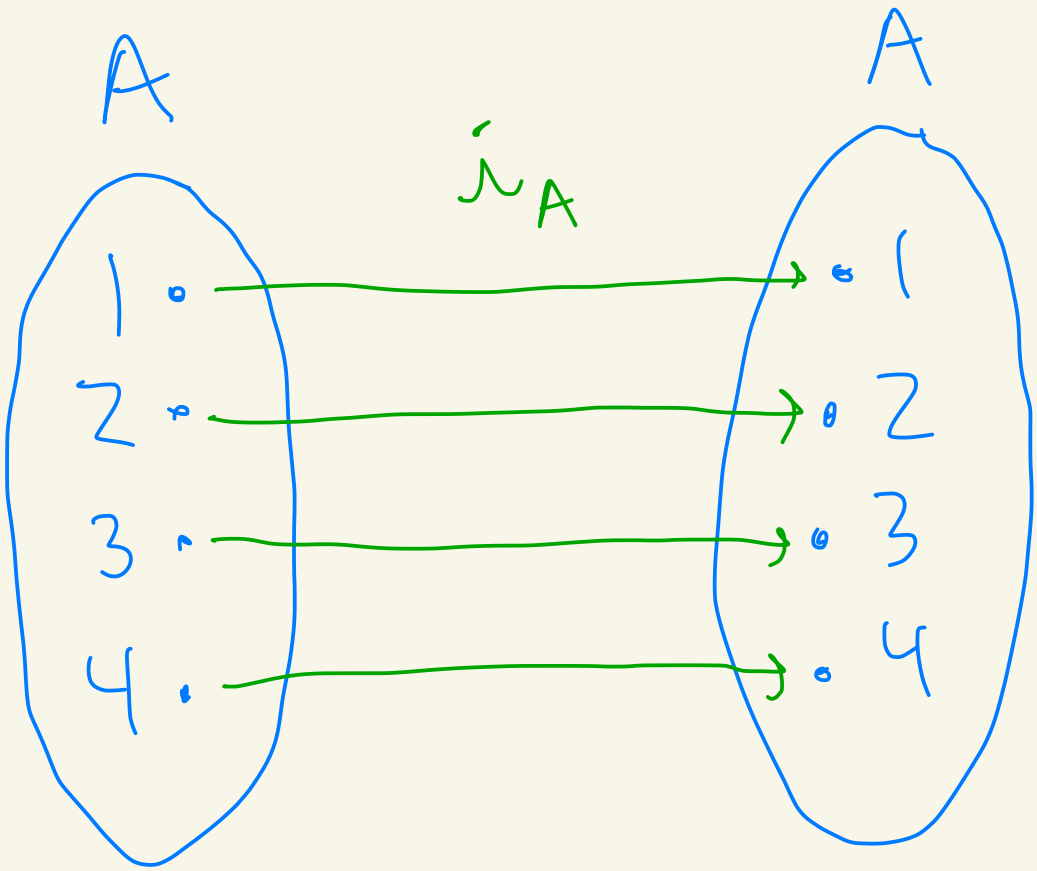
Sometimes we will just write i instead of i_A.

Formally you can think of

$$i_A = \{ (x, x) \mid x \in A \} \subseteq A \times A$$

$$\boxed{i_A(x) = x}$$

Ex: $A = \{1, 2, 3, 4\}$



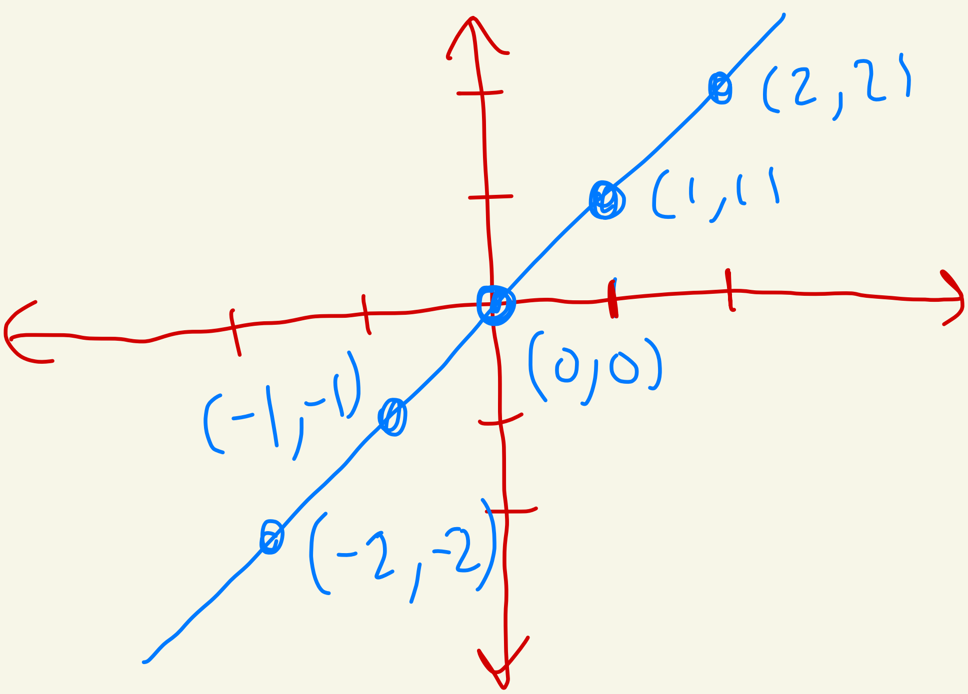
$i_A(1) = 1$

$i_A(2) = 2$

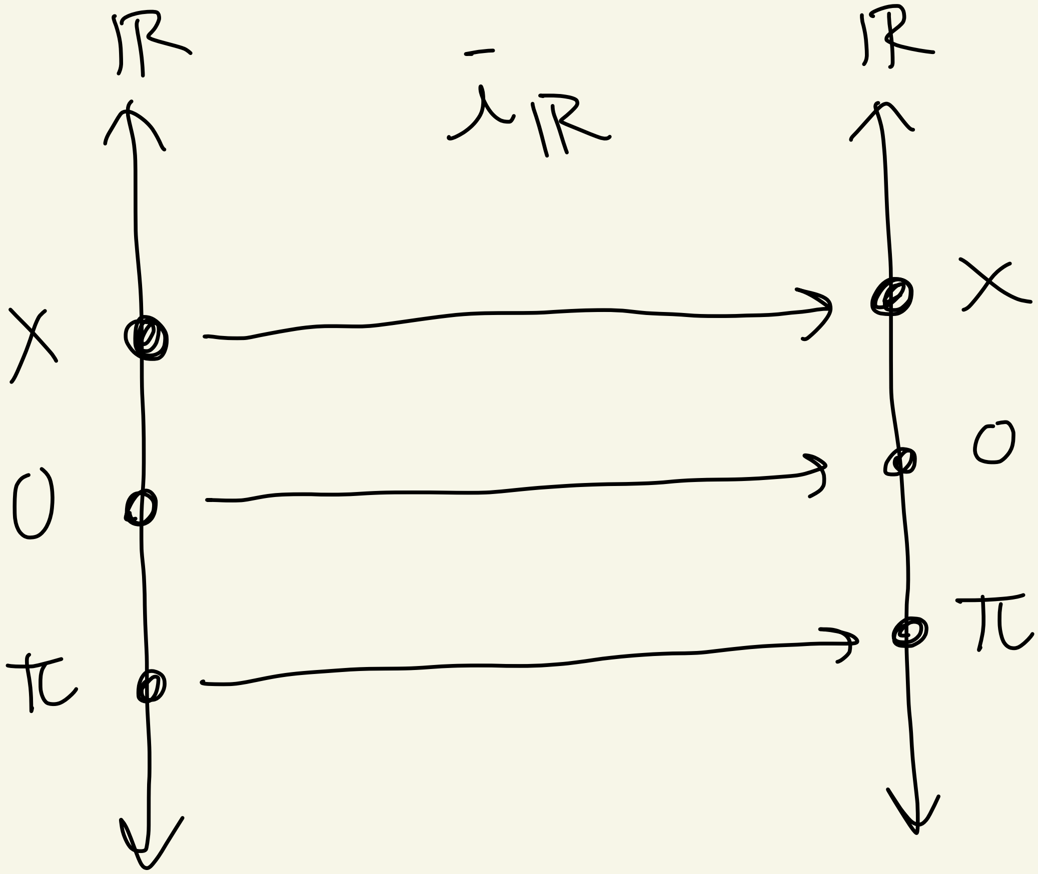
$i_A(3) = 3$

$i_A(4) = 4$

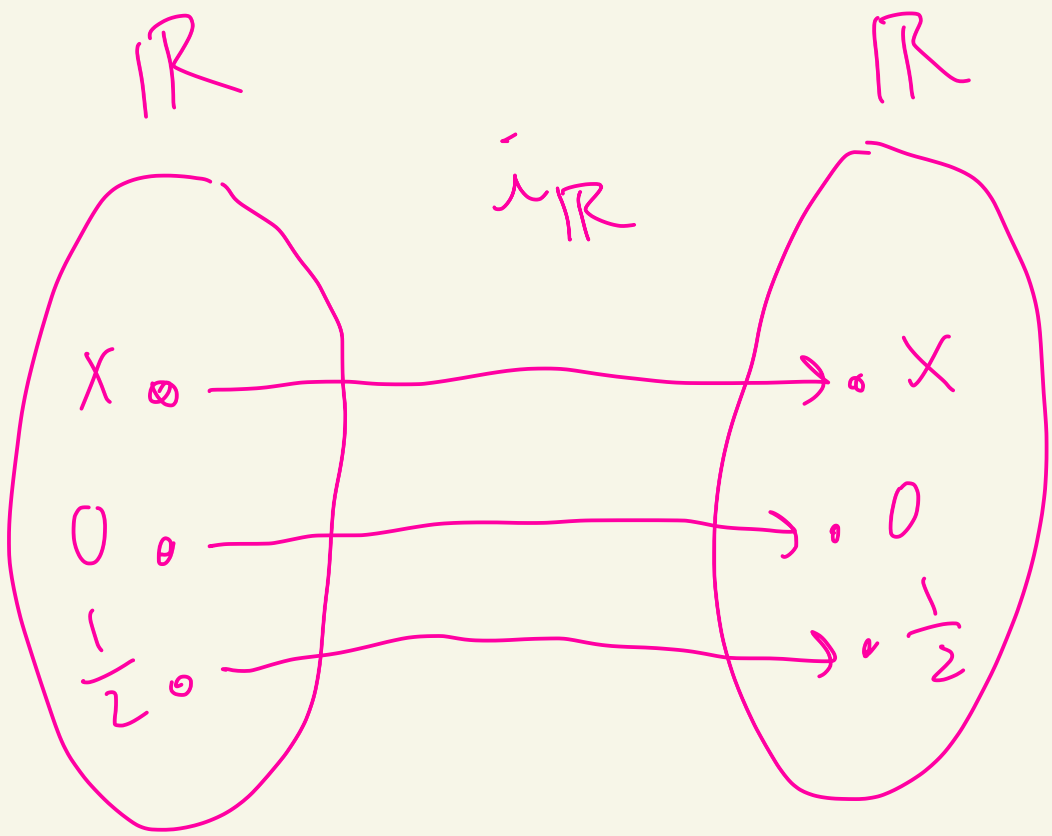
Ex: $A = \mathbb{R}$, $i_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$, $i_{\mathbb{R}}(x) = x$



graph way to draw $i_{\mathbb{R}}$



another way to draw



another way to draw

Ex: Let $n \in \mathbb{Z}$, $n \geq 2$.

Define the reduction modulo n map to be

map is another name for function some use mapping

$$\pi_n : \mathbb{Z} \longrightarrow \mathbb{Z}_n$$

where $\pi_n(x) = \overline{x}$

Ex: $n = 3$

$$\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$$

$$\pi_3 : \mathbb{Z} \to \mathbb{Z}_3, \quad \pi_3(x) = \overline{x}$$

Some computations are:

$$\pi_3(0) = \overline{0}$$

$$\pi_3(1) = \overline{1}$$

$$\pi_3(-1) = \overline{-1} = \overline{2}$$

$$\pi_3(-2) = \overline{-2} = \overline{1}$$

$\pi_3(2) = \overline{2}$

$\pi_3(3) = \overline{3} = \overline{0}$

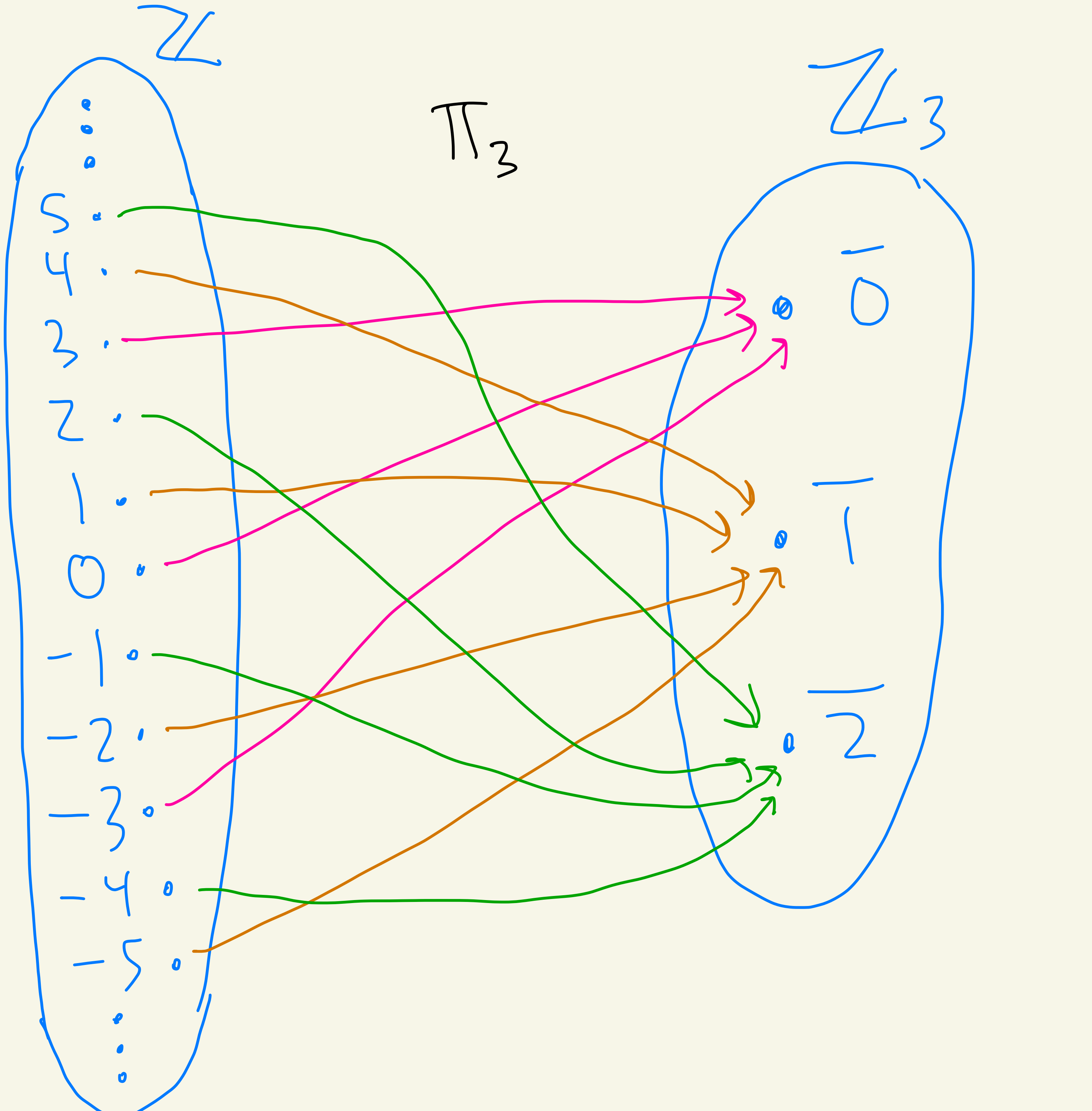
$\pi_3(4) = \overline{4} = \overline{1}$

$\pi_3(5) = \overline{5} = \overline{2}$

$\pi_3(-3) = \overline{-3} = \overline{0}$

$\pi_3(-4) = \overline{-4} = \overline{2}$

$\pi_3(-5) = \overline{-5} = \overline{1}$



$\text{domain}(\pi_3) = \mathbb{Z}$

$\text{co-domain}(\pi_3) = \mathbb{Z}_3$

$\text{range}(\pi_3) = \{\bar{0}, \bar{1}, \bar{2}\} = \mathbb{Z}_3$