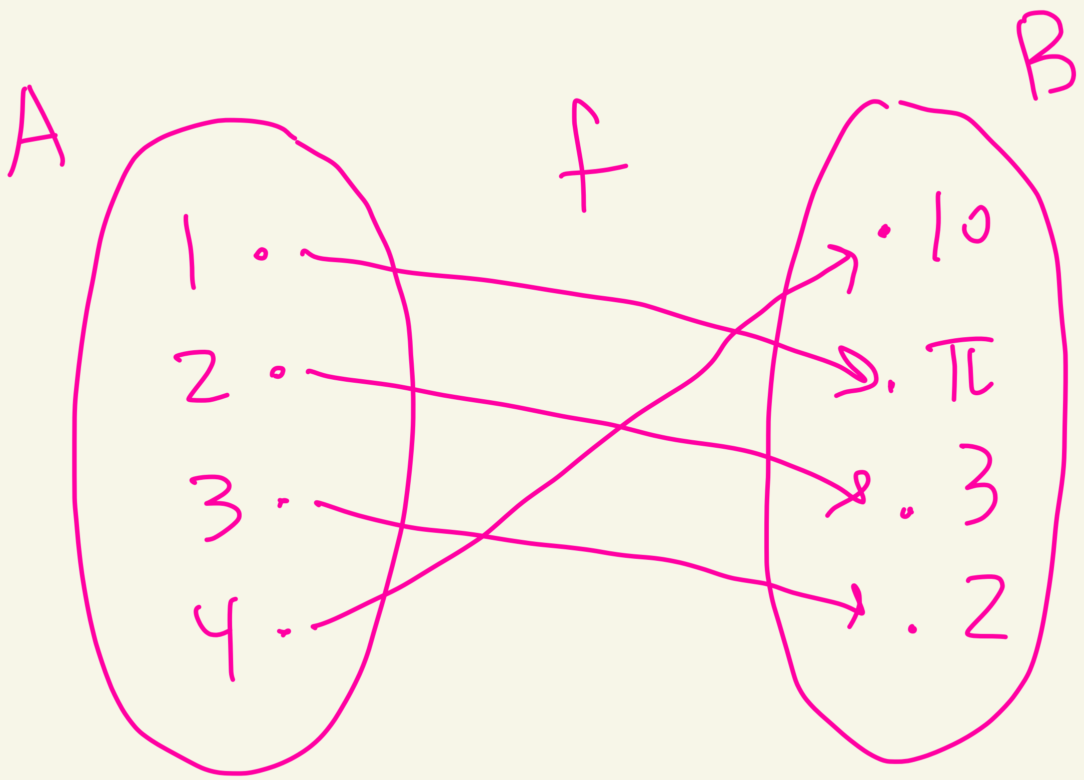
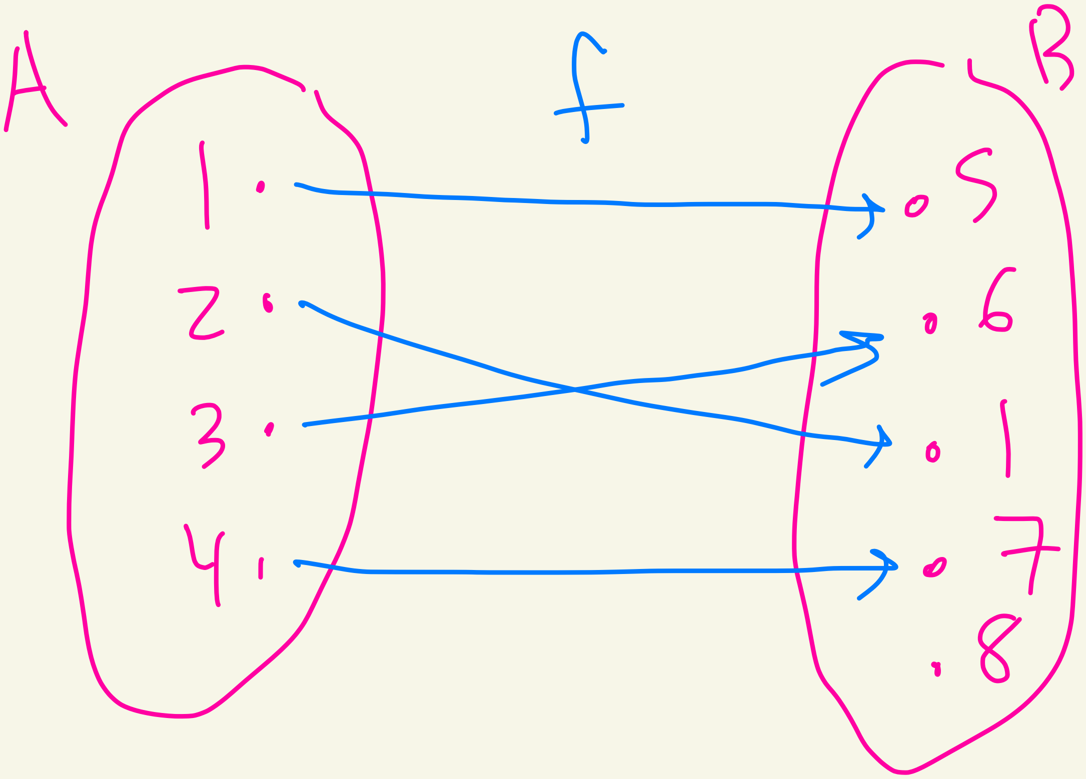
Math 3450
3/21/24

Def: Let A and B be sets and $f: A \to B$.
We say that f is a **bijection** if f is one-to-one and onto B.

Ex:



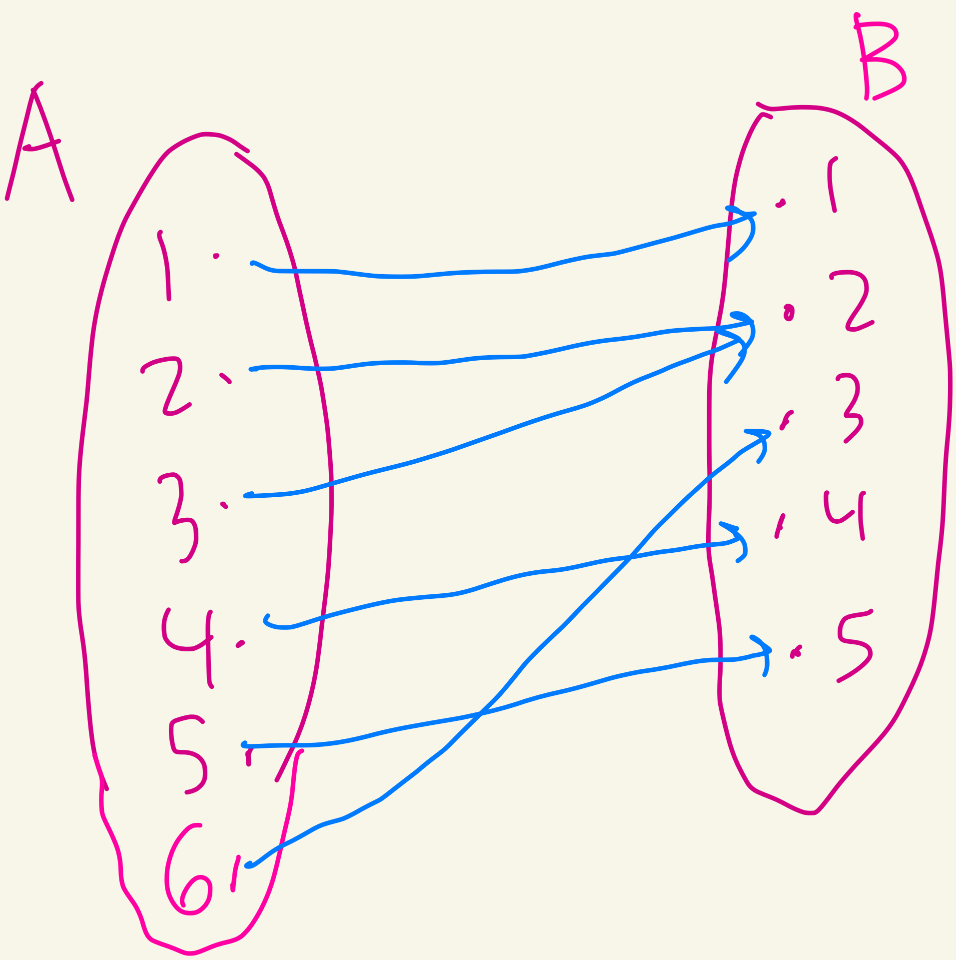
1-1 ✓
onto ✓
f is a bijection



1-1 ✓

onto B ✗

f is not a bijection



1-1 ✗

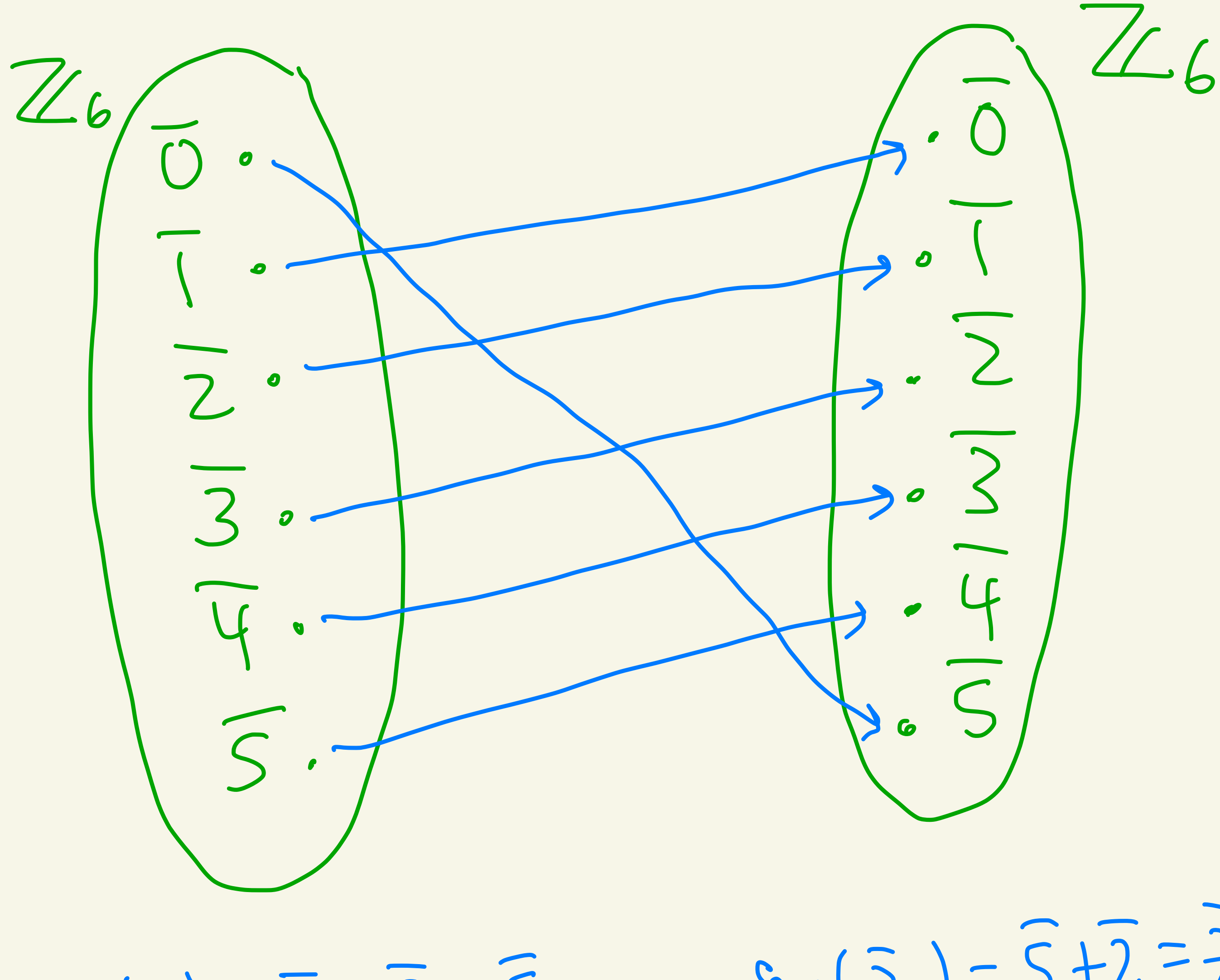
onto B ✓

f is not a bijection

Ex: (from Homework)

Given $a \in \mathbb{Z}$, define $g_a : \mathbb{Z}_n \to \mathbb{Z}_n$ by $g_a(\overline{x}) = \overline{x} + \overline{a}$

$\boxed{\text{Ex:}}$ $g_5 : \mathbb{Z}_6 \to \mathbb{Z}_6$, $g_5(\overline{x}) = \overline{x} + \overline{5}$



$g_5(\overline{0}) = \overline{0} + \overline{5} = \overline{5}$

$g_5(\overline{1}) = \overline{1} + \overline{5} = \overline{6} = \overline{0}$

$g_5(\overline{2}) = \overline{5} + \overline{2} = \overline{7} = \overline{1}$

$\vdots$

In the HW you show g_a is well-defined.

$\boxed{\text{Claim: Given } a \in \mathbb{Z}\text{, the function } g_a : \mathbb{Z}_n \to \mathbb{Z}_n \text{ given by } g_a(\overline{x}) = \overline{x} + \overline{a} \text{ is a bijection}}$

proof:

(one-to-one)

Suppose $g_a(\overline{x_1}) = g_a(\overline{x_2})$ where $\overline{x_1}, \overline{x_2} \in \mathbb{Z}_n$.

Then, $\overline{x_1} + \overline{a} = \overline{x_2} + \overline{a}$.

Then, $(\overline{x_1} + \overline{a}) + \overline{-a} = (\overline{x_2} + \overline{a}) + \overline{-a}$.

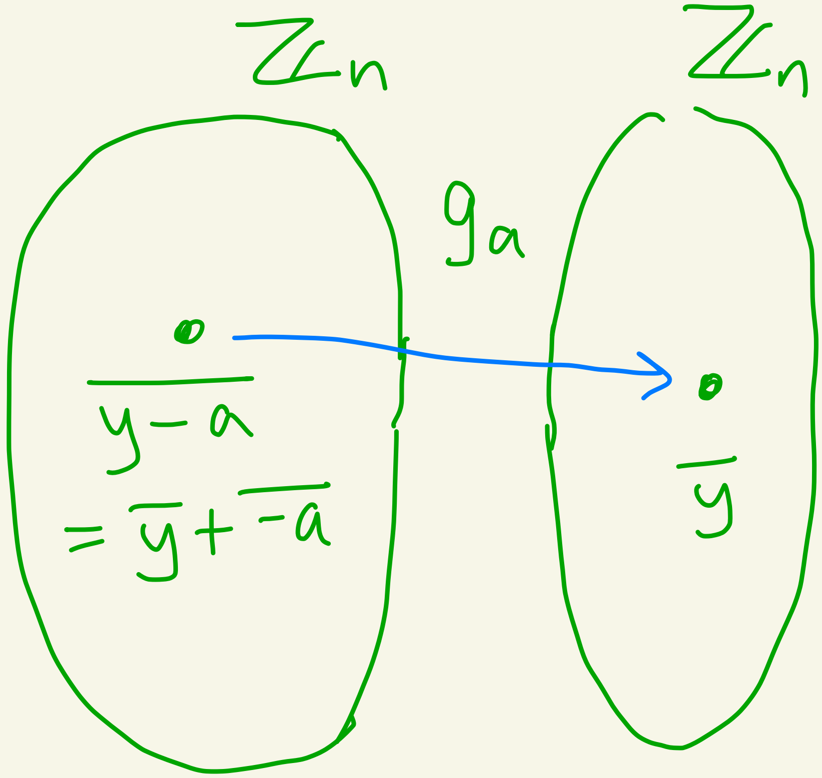
Thus, $\overline{x_1} + \overline{0} = \overline{x_2} + \overline{0}$.

So, $\overline{x_1} = \overline{x_2}$.

Thus, g_a is one-to-one.

(onto)

Let $\overline{y} \in \mathbb{Z}_n$, where $y \in \mathbb{Z}$.



Then,

$\overline{y-a} \in \mathbb{Z}_n$

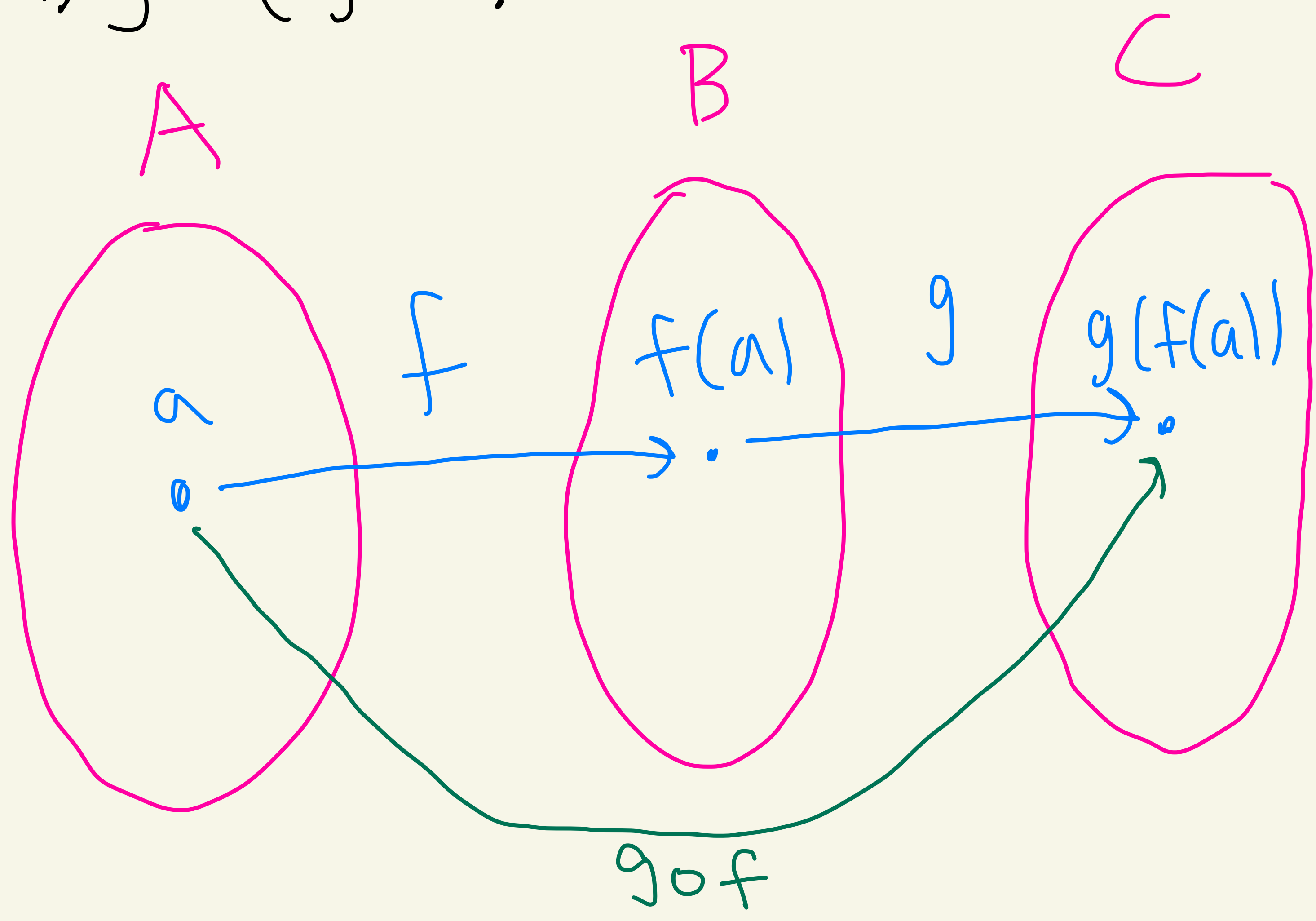
because $y-a \in \mathbb{Z}$.

And,

$$g_a(\overline{y-a}) = \overline{y-a} + \overline{a}$$
$$= \overline{y-a+a}$$
$$= \overline{y}$$

So, g_a is onto. $\blacksquare$

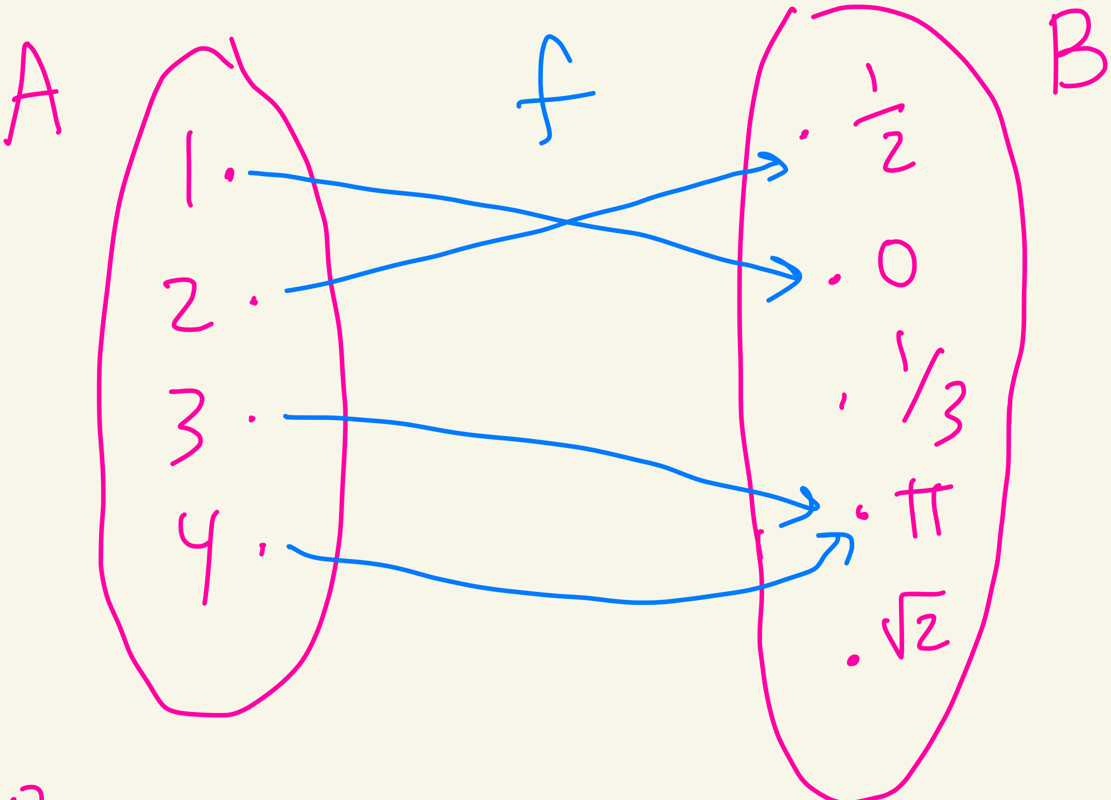
Def: Let A, B, C be sets. Let $f : A \to B$ and $g : B \to C$. Define the **composition** of f and g to be the function $(g \circ f) : A \to C$ by $(g \circ f)(a) = g(f(a))$

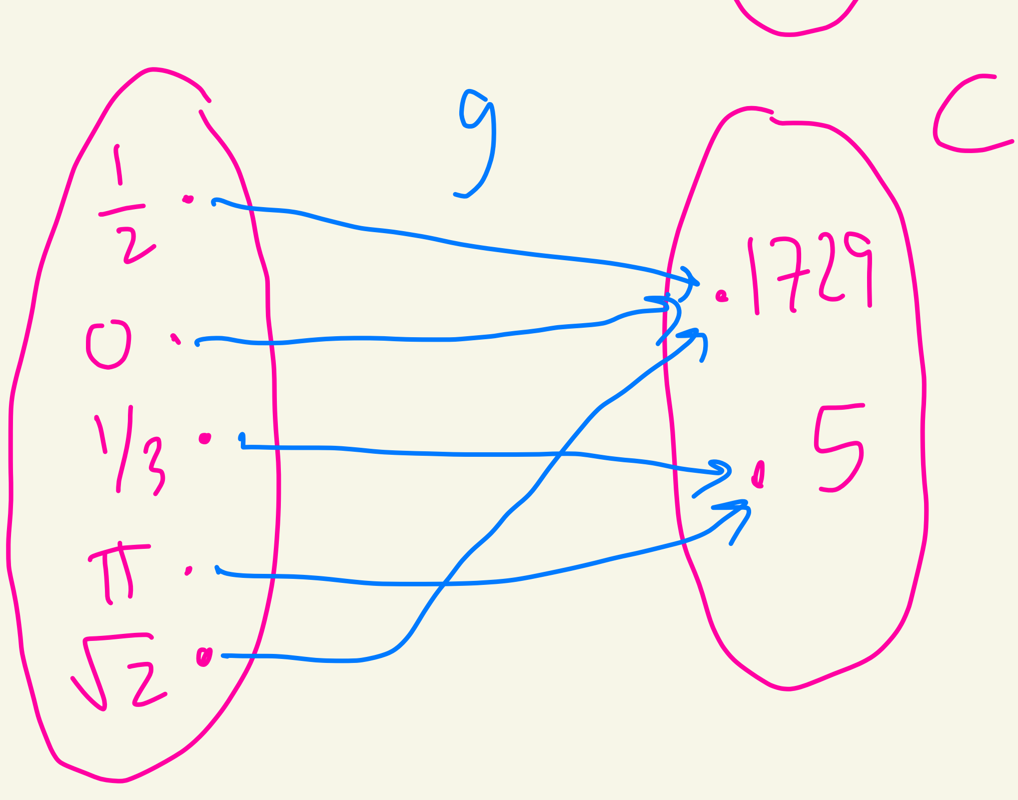


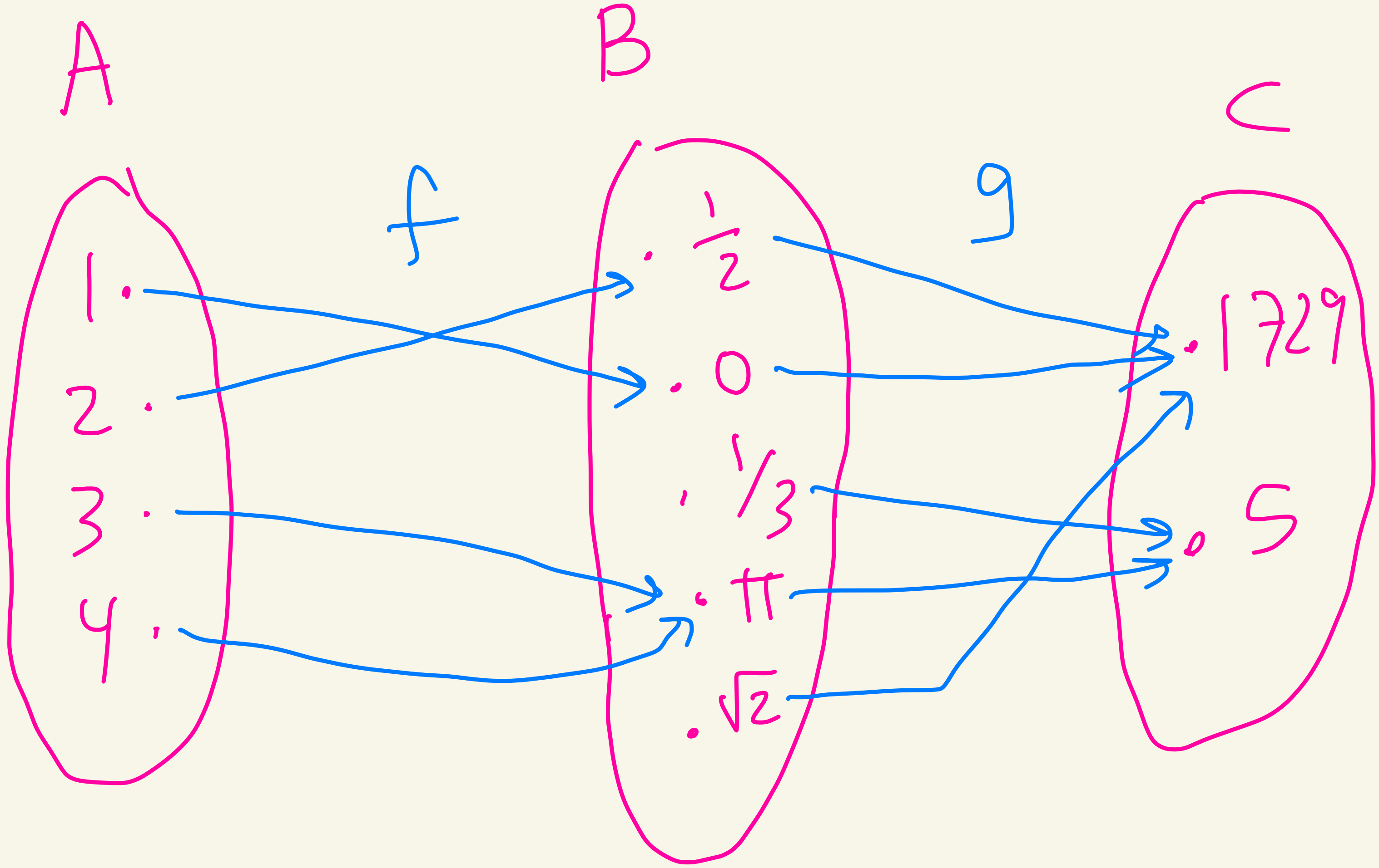
Ex: $A = \{1, 2, 3, 4\}$

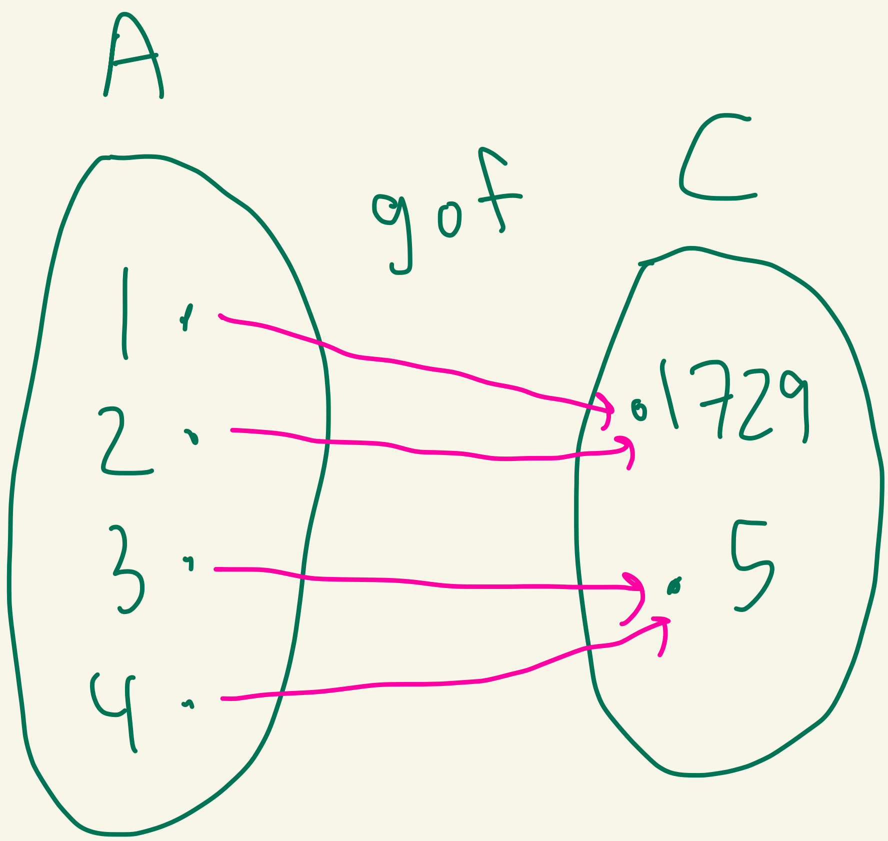
$B = \{\frac{1}{2}, 0, \frac{1}{3}, \pi, \sqrt{2}\}$

$C = \{1729, 5\}$









$$(g \circ f)(1) = g(f(1))$$
$$= g(0)$$
$$= 1729$$

$$(g \circ f)(2) = g(f(2))$$
$$= g\left(\tfrac{1}{2}\right)$$
$$= 1729$$

<u>Note:</u> $g \circ f$ is onto but not one-to-one

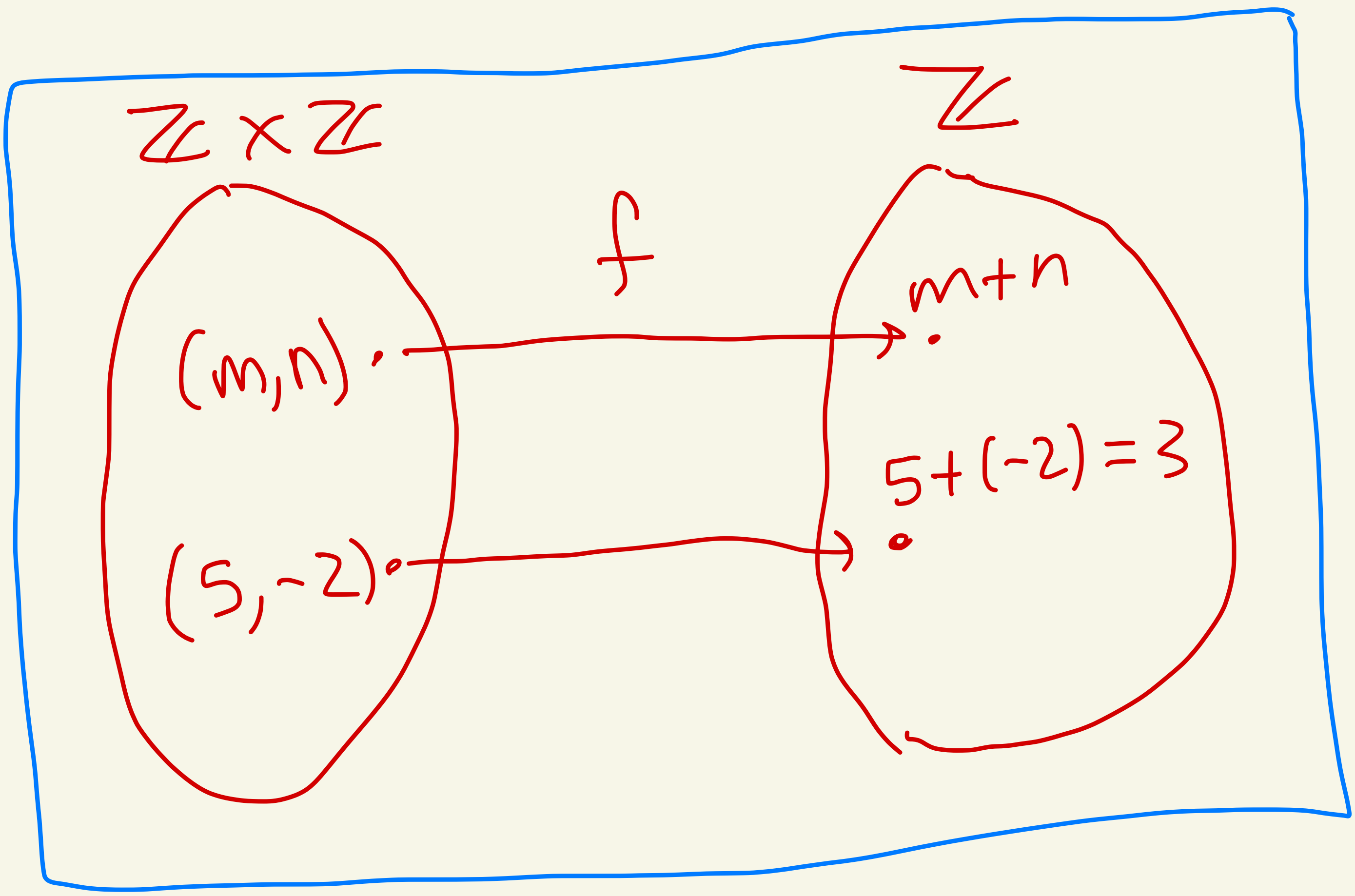
Ex: (Hammock 12.4 #9)

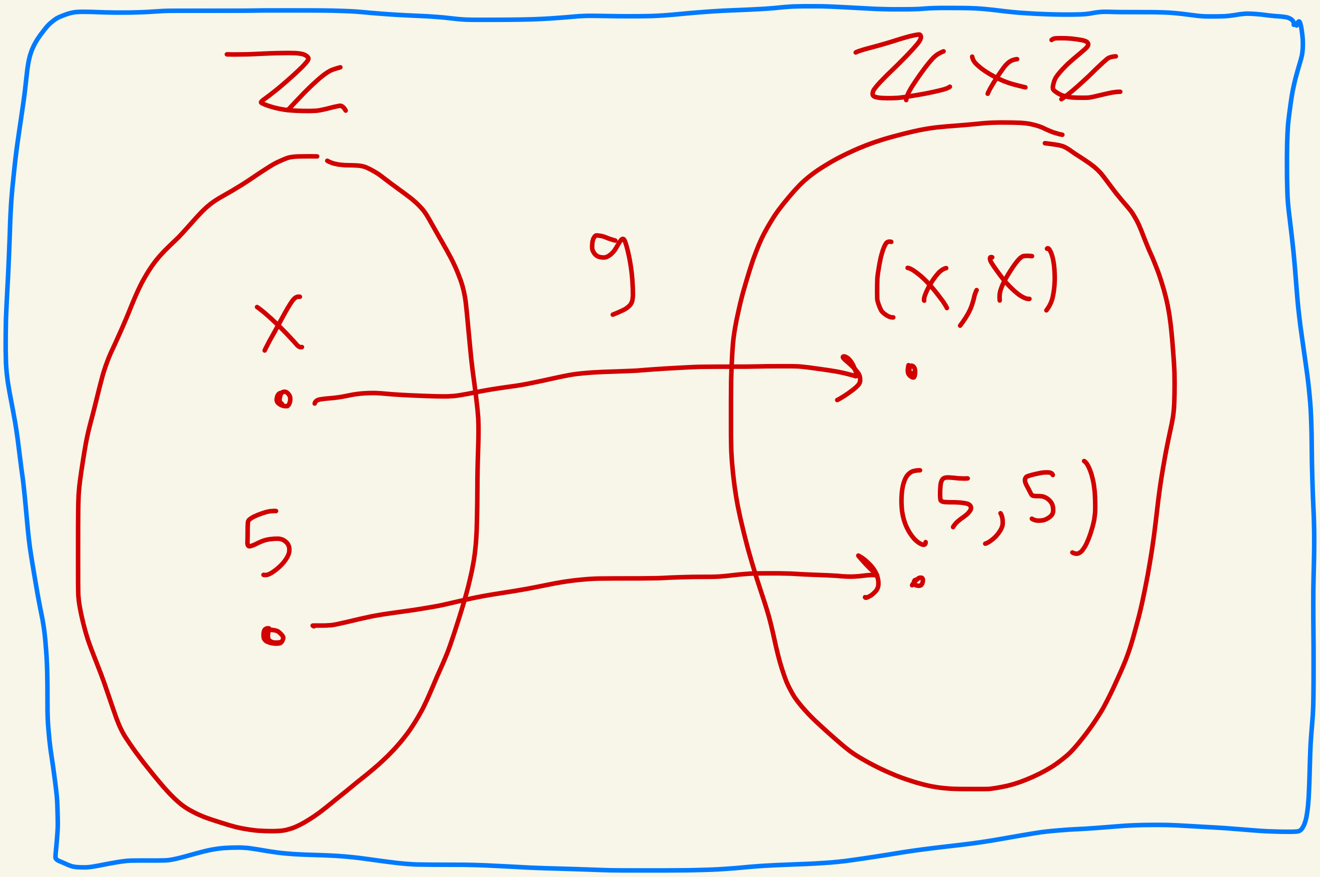
Define $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where

$$f(m,n) = m+n$$

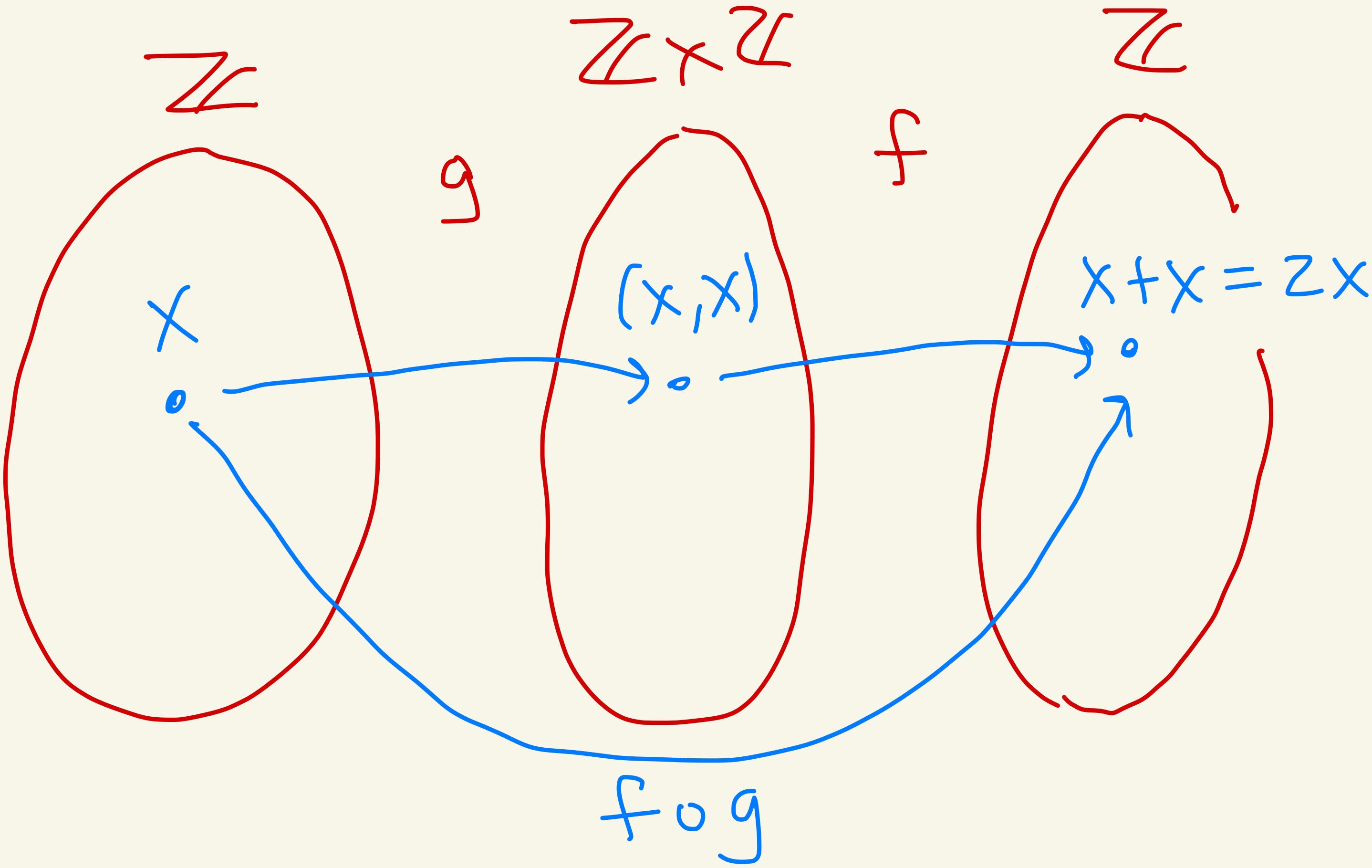
and $g: \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where

$$g(x) = (x, x)$$





Find formulas for $f \circ g$ and $g \circ f$

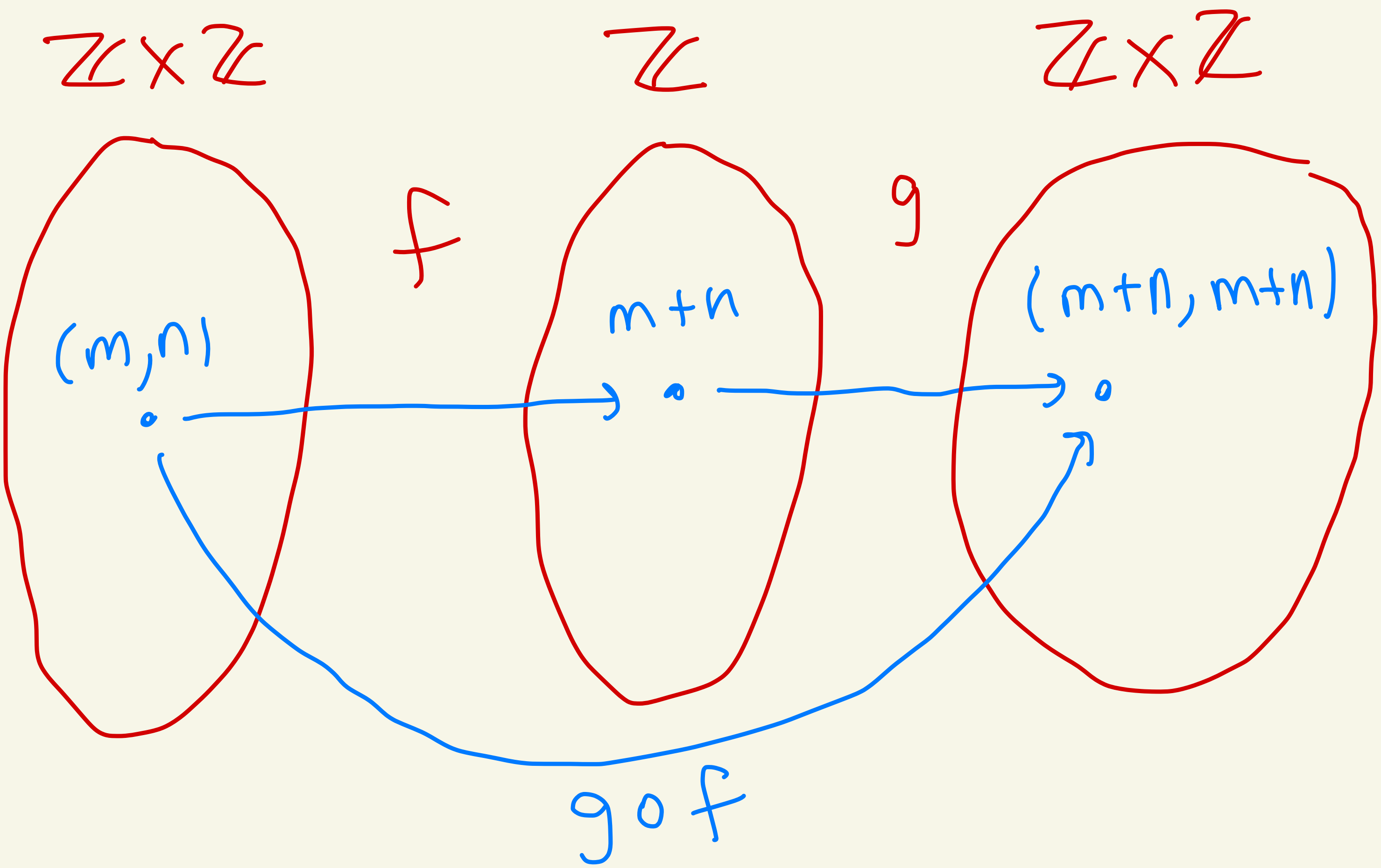


$f \circ g : \mathbb{Z} \to \mathbb{Z}$

$(f \circ g)(x) = f(g(x))$

$= f(x, x) = x + x = 2x$

So, $(f \circ g)(x) = 2x$



$g \circ f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$

$(g \circ f)(m, n) = g(f(m, n))$

$$= g(m+n)$$
$$= (m+n, m+n)$$

So, $(g \circ f)(m,n) = (m+n, m+n)$

Question: Is g 1-1?
Is g onto?

Claim: g is 1-1

pf: Suppose $g(x_1) = g(x_2)$ where $x_1, x_2 \in \mathbb{Z}$.

$g(x) = (x,x)$

Then, $(x_1, x_1) = (x_2, x_2)$

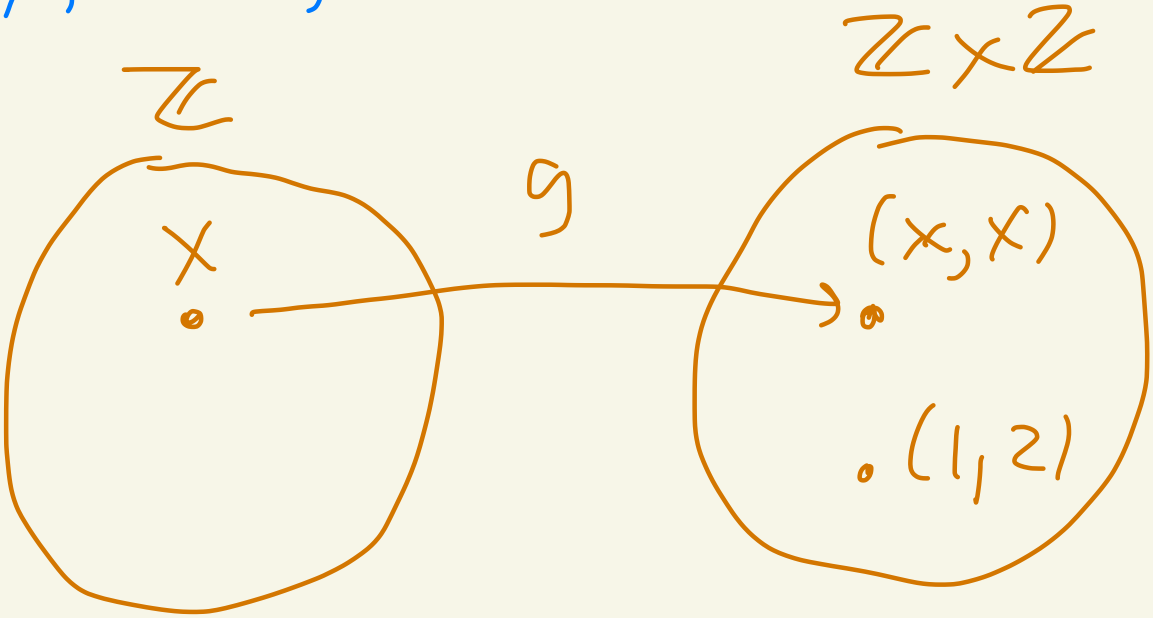
So, $x_1 = x_2$. $\blacksquare$

Claim: g is not onto.

Pf: Let $(1,2) \in \mathbb{Z} \times \mathbb{Z}$.

There is no $x \in \mathbb{Z}$ with

$g(x) = (x,x) = (1,2)$.



So, $(1,2) \notin \text{range}(g)$

So, g is not onto.