Math 3450

2/29/24

Def: A partition of a set S is a family of sets $\mathcal{A}$ where

1. every $A \in \mathcal{A}$ satisfies $A \subseteq S$,
2. $\displaystyle\bigcup_{A \in \mathcal{A}} A = S$
3. If $A, B \in \mathcal{A}$ and $A \neq B$, then $A \cap B = \phi$.

Ex: $S = \{1, 2, 3, 4, 5, 6\}$

$$\mathcal{A} = \{ \underbrace{\{1,3,5\}}_{A_1}, \underbrace{\{2,6\}}_{A_2}, \underbrace{\{4\}}_{A_3} \}$$

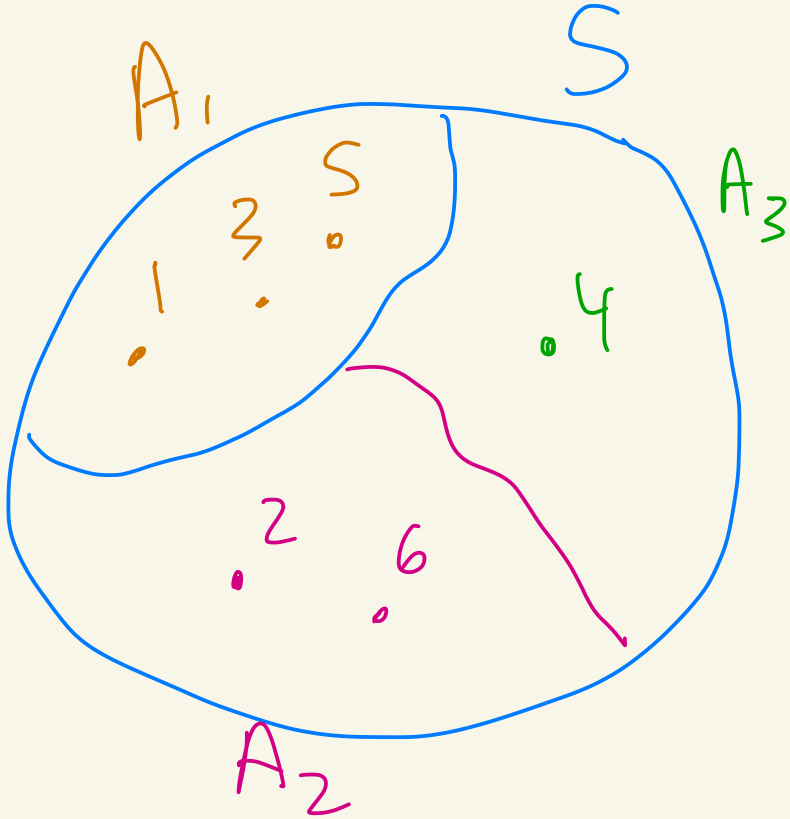
1. $A_1 \subseteq S$, $A_2 \subseteq S$, $A_3 \subseteq S$
2. $\displaystyle\bigcup_{A \in \mathcal{A}} A = A_1 \cup A_2 \cup A_3 = S$

(3) $A_1 \cap A_2 = \phi$

$A_1 \cap A_3 = \phi$

$A_2 \cap A_3 = \phi$

Thus, A is a partition of S



Ex:

$S = \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Consider the equivalence classes modulo $\boxed{n = 3}$. They are

$\overline{0} = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$

$\overline{1} = \{\ldots, -8, -5, -2, 1, 4, 7, \ldots\}$

$\overline{2} = \{\ldots, -7, -4, -1, 2, 5, 8, \ldots\}$

The set of equivalence classes is a partion of $\mathbb{Z}$.

$$A = \mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$$

<u>Theorem</u> Let S be a non-empty set. Let $\sim$ be an equivalence relation on S. Then the set of equivalence classes

$$S/_{\sim} = \{ \bar{a} \mid a \in S \}$$

is a partition of S.

<u>Ex:</u> when $\sim$ is mod 3 then $S/_{\sim} = \mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$

<u>proof:</u>

1. Let $\bar{a} \in S/_{\sim}$ where $a \in S$.

Then,

$$\bar{a} = \{ b \mid b \in S \text{ where } a \sim b \} \subseteq S$$

(2) We have that

$\boxed{(1)\ \bar{a} \subseteq S}$

$$S = \bigcup_{a \in S} \{a\} \subseteq \bigcup_{a \in S} \bar{a} = \bigcup_{\bar{a} \in S/\sim} \bar{a} \subseteq S$$

$\boxed{\text{super duper thm } a \in \bar{a}}$

$\boxed{S/\sim = \{ \bar{a} \mid a \in S \}}$

Thus, $S = \bigcup_{\bar{a} \in S/\sim} \bar{a}$.

(3) By the super-duper equivalence class theorem, if $a, b \in S$ and $\bar{a} \neq \bar{b}$, then $\bar{a} \cap \bar{b} = \phi$. $\blacksquare$

Theorem

Let S be a non-empty set.
Let $\mathcal{A}$ be a partition of S.
Define a relation $\sim$ on S by the following:

Given $a, b \in S$, then $a \sim b$ if and only if there exists $A \in \mathcal{A}$ where $a \in A$ and $b \in A$.

Then:

1. $\sim$ is an equivalence relation on S
2. $S/\sim = \mathcal{A}$

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proof: See notes from F19 - 10/2 $\blacksquare$

$\boxed{\text{HW 3}}$

3. $S = \mathbb{Z}$

$x \sim y$ means $2 \mid (x+y)$

(a) List 3 integers related to $x=4$

$2 \mid (4+0) \longrightarrow 4 \sim 0$

$2 \mid (4+2) \longrightarrow 4 \sim 2$

$2 \mid (\underbrace{4+(-10)}_{-6}) \longrightarrow 4 \sim (-10)$

(b) Prove $\sim$ is an equivalence relation on $S = \mathbb{Z}$.

$\boxed{\text{(reflexive)}}$ Let $a \in \mathbb{Z}$.

Then,

$$a+a=2a$$

So, $2 \mid (a+a)$.

Thus, $a \sim a$.

(symmetric) Let $a, b \in \mathbb{Z}$ where $a \sim b$.

Since $a \sim b$ we know $2 \mid (a+b)$.

Thus, $2 \mid (b+a)$.

Hence, $b \sim a$.

(transitive) Let $a, b, c \in \mathbb{Z}$ where $a \sim b$ and $b \sim c$.

This gives $2 \mid (a+b)$ and $2 \mid (b+c)$.

Thus, $a+b=2k$ and $b+c=2\ell$

where $k, l \in \mathbb{Z}$.

It follows that

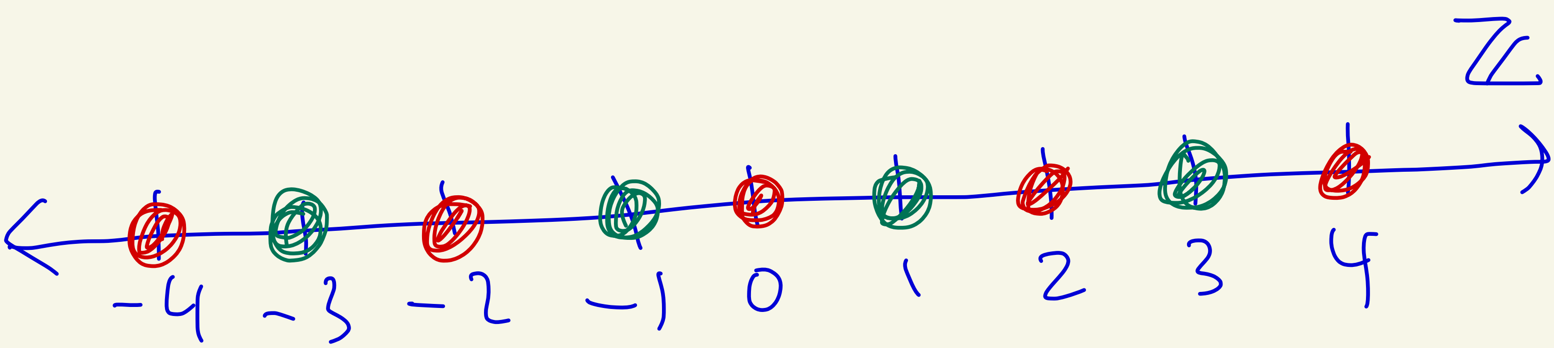
$$a+c = (2k-b)+(2l-b)$$
$$= 2k+2l-2b$$
$$= 2\underbrace{(k+l-b)}_{\text{this is in } \mathbb{Z} \text{ since } k,l,b \in \mathbb{Z}}$$

Thus, $2 \mid (a+c)$.

So, $a \sim c$.

$\boxed{(b)}$

(c/d) Find the equivalence classes



$$\overline{0} = \{ y \mid y \in \mathbb{Z} \text{ where } \underbrace{2 \mid (0+y)}_{0 \sim y} \}$$

$$= \{ y \mid y \in \mathbb{Z} \text{ where } 2 \mid y \}$$

$$\overline{1} = \{ y \mid y \in \mathbb{Z} \text{ where } \underbrace{2 \mid (1+y)}_{1 \sim y} \}$$

$$= \{ \ldots, -5, -3, -1, 1, 3, 5, 7, \ldots \}$$

HW 2

(14) (b) Let A and B be sets.
Prove: $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

proof:

Let $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$.
So either $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$.
Thus, either $X \subseteq A$ or $X \subseteq B$.
This implies that either
$X \subseteq A \subseteq A \cup B$ or $X \subseteq B \subseteq A \cup B$.

Thus, $X \subseteq A \cup B$.
So, $X \in \mathcal{P}(A \cup B)$.

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