Math 3450 - Homework # 4
Functions

Part 1 - One-to-one / Onto / Inverse

---

1. Consider the following functions. For each function f, (i) either prove that f is one-to-one or give an example to show otherwise, and (ii) either prove that f is onto, or give an example to show otherwise. (iii) If f is a bijection, find a formula for f^{-1}.

   (a) $f : \mathbb{Q} \to \mathbb{Q}$ where $f(x) = x^3$.

   (b) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x + 5$.

   (c) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^4 - 16$.

   (d) $f : \mathbb{Z}_4 \to \mathbb{Z}_4$ given by $f(\overline{x}) = \overline{2} \cdot \overline{x} + \overline{1}$.

   (e) $f : \mathbb{Z}_4 \to \mathbb{Z}_4$ given by $f(\overline{x}) = \overline{3} \cdot \overline{x} + \overline{1}$.

   (f) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $f(a, b) = a + b$.

   (g) $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $g(m, n) = (2m + 1, n)$.

   (h) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $f(a, b) = (a + b, a - b)$.

   (i) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $f(m, n) = (5m + 4n, 4m + 3n)$.

2. Suppose that $f : A \to B$ and $g : B \to C$. Prove: If $g \circ f$ is onto, then g is onto.

3. Suppose that $f : A \to B$ and $g : B \to C$. Prove: If f is not one-to-one, then $g \circ f$ is not one-to-one.

Part 2 - Functions applied to sets

---

4. Suppose that A, B, W, Z be sets where $W \subseteq A$ and $Z \subseteq A$. Let $f : A \to B$.

   (a) Prove that $f(W \cup Z) = f(W) \cup f(Z)$.

(b) Prove that $f(W \cap Z) \subseteq f(W) \cap f(Z)$.

(c) Give an example to show that $f(W \cap Z) = f(W) \cap f(Z)$ is not always true.

(d) Prove that if $W \subseteq Z$, then $f(W) \subseteq f(Z)$.

5. Suppose that A, B, W, Z be sets where $W \subseteq B$ and $Z \subseteq B$. Let $f : A \to B$.

   (a) Prove that $f^{-1}(W \cap Z) = f^{-1}(W) \cap f^{-1}(Z)$.

   (b) Prove that $f^{-1}(W \cup Z) = f^{-1}(W) \cup f^{-1}(Z)$.

   (c) Prove that $A - f^{-1}(W) = f^{-1}(B - W)$.

   (d) Prove that if $W \subseteq Z$, then $f^{-1}(W) \subseteq f^{-1}(Z)$.

Part 3 - Various functions illustrating all the concepts

---

6. Consider the function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $f(\overline{x}) = \overline{x}^2$.

   (a) Draw a picture of f when $n = 5$.

   (b) Draw a picture of f when $n = 6$.

   (c) Prove that f a well-defined function.

   (d) Prove that if $n > 2$ then f is not one-to-one.

7. Let n be an integer with $n \geq 2$. Let a be an integer. Define $g_a : \mathbb{Z}_n \to \mathbb{Z}_n$ by the formula $g_a(\overline{x}) = \overline{x} + \overline{a}$.

   (a) Draw a picture of g_3 and g_2 when $n = 4$.

   (b) Compute and draw a picture of $g_3 \circ g_2$ and $g_2 \circ g_3$ when $n = 4$.

   (c) Prove that g_a is well-defined.

   (d) Prove that g_a is a bijection for any n.

   (e) Find a formula for g_a^{-1}.

8. Let $n \geq 2$ be an integer. Consider the reduction mod n map $\pi_n : \mathbb{Z} \to \mathbb{Z}_n$ given by the formula $\pi_n(x) = \overline{x}$.

   For example, $\pi_6(2) = \overline{2}$ and $\pi_6(18) = \overline{18} = \overline{0}$ since $18 \equiv 0 (\text{mod } 6)$.

(a) Calculate $\pi_6(-1)$, $\pi_6(10)$, $\pi_6(7)$, and $\pi_6(-17)$. Draw a picture of the π_6 map. Is π_6 one-to-one? Is π_6 onto?

(b) Let $X = \{1, 17, -5, 102, -13\}$. Calculate $\pi_6(X)$.

(c) Let $Y = \{\overline{0}\}$. Prove that $\pi_6^{-1}(Y) = \{6k \mid k \in \mathbb{Z}\}$.

(d) Let $Y = \{\overline{1}\}$. Prove that $\pi_6^{-1}(Y) = \{6k + 1 \mid k \in \mathbb{Z}\}$.

(e) What is $\pi_6^{-1}(\{\overline{0}, \overline{3}\})$ equal to? Prove your answer.

9. Let a and n be integers with $n \geq 2$. Define $f_a : \mathbb{Z}_n \to \mathbb{Z}_n$ by $f_a(\overline{x}) = \overline{a} \cdot \overline{x}$.

   (a) Draw a picture of f_4 when $n = 6$.

   (b) Draw a picture of f_2 when $n = 3$.

   (c) Prove that f_a is a well-defined function.

   (d) Prove that $f_c \circ f_d = f_{cd}$.

   (e) Prove that $f_{cd} = f_{dc}$ for all integers c and d.

   (f) Prove: If $y \equiv w (\text{mod } n)$, then $f_y = f_w$.

   (g) Prove that if $\gcd(a, n) > 1$, then f_a is not a bijection. [Hint: Note that $f_a(\overline{0}) = \overline{0}$. Find $\overline{k} \neq \overline{0}$ with $f_a(\overline{k}) = \overline{0}$.]

   (h) Consider $f_3 : \mathbb{Z}_5 \to \mathbb{Z}_5$. Find f_3^{-1} and express it in the form f_b for some integer b.

   (i) Prove: If there exists $\overline{b} \in \mathbb{Z}_n$ with $\overline{b} \cdot \overline{a} = 1$, then f_a is a bijection. [Note: In Math 4460 you will show that there exists such a $\overline{b}$ if and only if $\gcd(a, n) = 1$.]

10. Let A be a set. Define the function $f : \mathcal{P}(A) \to \mathcal{P}(A)$ where $f(X) = A - X$ for any $X \subseteq A$.

   (a) Draw a picture of f when $A = \{1, 2, 3\}$.

   (b) If $X \subseteq A$, then $A - (A - X) = X$.

   (c) For general A prove that f is a bijection.

   (d) For general A prove that $f = f^{-1}$.

11. Let $A = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, 4, 5, 6, 7, \ldots\}$. Let $f : A \times A \to A$ where $f(m, n) = m^2 + n^2$.

   (a) Calculate $f(3, 5)$, $f(1, 1)$, and $f(2, 1)$.

(b) Let $C = \{(0,0), (1,10), (2,5)\}$. Calculate $f(C)$.

(c) Let $B = \{1,2,3,4\}$. Find $f^{-1}(B)$.

(d) Show that f is not one-to-one.

(e) Show that f is not onto.