Math 3450 - Homework # 2 - Part A
Equivalence Relations

1. A set S and a relation $\sim$ on S is given. For each example, check if $\sim$ is (i) reflexive, (ii) symmetric, and/or (iii) transitive. If $\sim$ satisfies the property that you are checking, then prove it. If $\sim$ does not satisfy the property that you are checking, then give an example to show it.

   (a) $S = \mathbb{R}$ where $a \sim b$ if and only if $a \leq b$.

   Solution:

   (i) Yes, $\sim$ is reflexive. Proof: Let $a \in \mathbb{R}$. Then $a \leq a$. So $a \sim a$.

   (ii) No, $\sim$ is not symmetric. Counterexample: $3 \leq 4$, but $4 \not\leq 3$. That is, $3 \sim 4$ but $4 \not\sim 3$.

   (iii) Yes, $\sim$ is transitive. Proof: Let $a, b, c \in \mathbb{R}$ and suppose that $a \sim b$ and $b \sim c$. Then $a \leq b$ and $b \leq c$. So $a \leq c$. Thus $a \sim c$.

   (b) $S = \mathbb{R}$ where $a \sim b$ if and only if $|a| = |b|$.

   Solution:

   (i) Yes, $\sim$ is reflexive. Proof: Let $a \in \mathbb{R}$. Then $|a| = |a|$. So $a \sim a$.

   (ii) Yes, $\sim$ is symmetric. Proof: Let $a, b \in \mathbb{R}$ and suppose that $a \sim b$. Then $|a| = |b|$. So $|b| = |a|$. Thus $b \sim a$.

   (iii) Yes, $\sim$ is transitive. Proof: Let $a, b, c \in \mathbb{R}$ and suppose that $a \sim b$ and $b \sim c$. Then $|a| = |b|$ and $|b| = |c|$. So $|a| = |c|$. Thus $a \sim c$.

   (c) $S = \mathbb{Z}$ where $a \sim b$ if and only if $a|b$.

   (i) Yes, $\sim$ is reflexive. Proof: Let $a \in \mathbb{Z}$. Then $a(1) = a$. Hence $a|a$. So $a \sim a$.

   (ii) No, $\sim$ is not symmetric. Counterexample: $3|6$, but $6 \nmid 3$.

   (iii) Yes, $\sim$ is transitive. Proof: Let $a, b, c \in \mathbb{Z}$. Suppose that $a \sim b$ and $b \sim c$. Then $a|b$ and $b|c$. Thus there exists $k, m \in \mathbb{Z}$ such that $ak = b$ and $bm = c$. Then $c = bm = (ak)m = a(km)$. So $a|c$. Thus $a \sim c$.

   (d) S is the set of subsets of $\mathbb{N}$ where $A \sim B$ if and only if $A \subseteq B$. Some examples of elements of S are $\{1, 10, 199\}$, $\{2, 7, 10\}$, and $\{2, 10, 3, 7\}$. Note that $\{2, 7, 10\} \sim \{2, 10, 3, 7\}$

   Solution:

(i) Yes, $\sim$ is reflexive. Proof: $A \subseteq A$ for all $A \in S$.

(ii) No, $\sim$ is not symmetric. Counterexample: $\{3\} \subseteq \{3, 42\}$, but $\{3, 42\} \not\subseteq \{3\}$.

(iii) Yes, $\sim$ is transitive. Proof: Let $A, B, C \in S$ with $A \sim B$ and $B \sim C$. Then $A \subseteq B$ and $B \subseteq C$. We want to show that $A \subseteq C$. Let $x \in A$. Since $A \subseteq B$, we have that $x \in B$. Since $B \subseteq C$ we have that $x \in C$. So $A \subseteq C$ and thus $A \sim C$.

2. Consider the set $S = \mathbb{R}$ where $x \sim y$ if and only if $x^2 = y^2$.

(a) Find all the numbers that are related to $x = 1$. Repeat this exercise for $x = \sqrt{2}$ and $x = 0$.

Solution:

$1 \sim 1$ since $1^2 = 1^2$. We also have $1 \sim (-1)$ since $1^2 = (-1)^2$. There are no other elements related to 1.

$\sqrt{2} \sim \sqrt{2}$ since $(\sqrt{2})^2 = (\sqrt{2})^2$. We also have $\sqrt{2} \sim (-\sqrt{2})$ since $(\sqrt{2})^2 = (-\sqrt{2})^2$. There are no other elements related to $\sqrt{2}$.

$0 \sim 0$ since $0^2 = 0^2$. There are no other elements related to 0.

(b) Prove that $\sim$ is an equivalence relation on S.

Solution:

Proof. Reflexive: We know that $x^2 = x^2$ for all real numbers x. Therefore $x \sim x$ for all real numbers x. So $\sim$ is reflexive.

Symmetric: Let $x, y \in \mathbb{R}$. Suppose that $x \sim y$.

Since $x \sim y$ we have that $x^2 = y^2$.

So $y^2 = x^2$.

Therefore $y \sim x$.

Transitive Let $x, y, z \in \mathbb{R}$. Suppose that $x \sim y$ and $y \sim z$.

Since $x \sim y$ we have that $x^2 = y^2$.

Since $y \sim z$ we have that $y^2 = z^2$.

So $x^2 = y^2 = z^2$.

Therefore $x \sim z$. □

(c) Draw a number line. Draw a picture of the equivalence class of 1. Repeat this for $x = 0$, $x = \sqrt{6}$, $x = -3$.

Solution: For the equivalence class of 1, draw the number line and circle the numbers $-1, 1$.

For the equivalence class of 0, draw the number line and circle the number 0.

For the equivalence class of $\sqrt{6}$, draw the number line and circle the numbers $-\sqrt{6}, \sqrt{6}$.

For the equivalence class of -3, draw the number line and circle the numbers $-3, 3$.

(d) Describe the elements of $S/\sim$.

Solution:

If $x \neq 0$, then the equivalence class of x is $\overline{x} = \{-x, x\}$. The equivalence class of 0 is $\overline{0} = \{0\}$.

3. Consider the set $S = \mathbb{Z}$ where $x \sim y$ if and only if $2|(x+y)$.

(a) List six numbers that are related to $x = 4$.

Solution:

$4 \sim (-4)$ since $2|(4 + (-4))$.
$4 \sim (-2)$ since $2|(4 + (-2))$.
$4 \sim (0)$ since $2|(4 + (0))$.
$4 \sim (2)$ since $2|(4 + (2))$.
$4 \sim (4)$ since $2|(4 + (4))$.
$4 \sim (6)$ since $2|(4 + (6))$.

(b) Prove that $\sim$ is an equivalence relation on S.

Proof. Reflexive: Let $x \in \mathbb{Z}$.
Since $2|2x$ we have that $2|(x + x)$.
So $x \sim x$.
Symmetric: Let $x, y \in \mathbb{Z}$ and suppose that $x \sim y$.
Thus $2|(x + y)$.
So $2|(y + x)$.
So $y \sim x$.
Transitive: Let $x, y, z \in \mathbb{Z}$ and suppose that $x \sim y$ and $y \sim z$.
Therefore $2|(x + y)$ and $2|(y + z)$.
So there exist $k, \ell \in \mathbb{Z}$ such that $2k = x + y$ and $2\ell = y + z$.

Add these equations to get $2k + 2\ell = x + 2y + z$.

Subtract $2y$ from both sides to get $2(k + \ell - y) = x + z$.

Note that $k + \ell - y \in \mathbb{Z}$, because $k, \ell, y \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under addition and subtraction.

So $2|(x + z)$.

So $x \sim z$.

□

(c) Draw a picture of the set of integers. Next, circle the numbers that are in the equivalence class of -3.

Solution: Draw a picture and circle these numbers:

$\ldots, -7, -5, -3, -1, 1, 3, 5, 7, \ldots$

(d) Describe the elements of $S/\sim$. Draw a picture of several equivalence classes.

Solution: Draw a picture of the following:

$$\begin{aligned} \overline{0} &= \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\} = \overline{-2} = \overline{2} = \overline{4} = \overline{-4} = \cdots \\ \overline{1} &= \{\ldots, -7, -5, -3, -1, 1, 3, 5, 7, \ldots\} = \overline{-1} = \overline{3} = \overline{-3} = \overline{-5} = \cdots \end{aligned}$$

So $S/\sim$ is equal to $\{\overline{0}, \overline{1}\}$. That is, one equivalence class is the set of all odd numbers; the other equivalence class is the set of all even numbers.

4. (Constructing the rational numbers from the integers) Let $S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Define the relation $\sim$ on S where $(a, b) \sim (c, d)$ if and only if $ad = bc$.

(a) Is $(1, 5) \sim (-3, -15)$?

Solution: Yes, because $1(-15) = 5(-3)$.

(b) Is $(-1, 1) \sim (2, 3)$?

Solution: No, because $(-1)(3) \neq 1(2)$.

(c) Prove that $\sim$ is an equivalence relation.

Proof. Reflexive: Let $(a, b) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$.

Then $ab = ba$.

So $(a, b) \sim (a, b)$.

Symmetric: Let $(a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$.
Suppose $(a, b) \sim (c, d)$.
We know that $ad = bc$, because $(a, b) \sim (c, d)$.
So $cb = da$.
Hence $(c, d) \sim (a, b)$.
Transitive: Let $(a, b), (c, d), (e, f) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$.
Note that $d \neq 0$ and $f \neq 0$ since $d, f \in \mathbb{Z} - \{0\}$.
Suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.
We know that $ad = bc$ and $cf = de$, because $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.
Thus

$$ad = bc = b\left(\frac{de}{f}\right) = \frac{bde}{f}$$

Thus $adf = bde$.
Since $d \neq 0$ we can divide by d to get $af = be$.
So $(a, b) \sim (e, f)$ since $af = be$.

Therefore, $\sim$ is an equivalence relation, because it is reflexive, symmetric, and transitive.

□

(d) List five elements from each of the following equivalence classes: $\overline{(1, 1)}$, $\overline{(0, 2)}$, $\overline{(2, 3)}$.
Solution: Some possible answers:
$(2, 2), (3, 3), (4, 4), (5, 5), (47, 47) \in \overline{(1, 1)}$.
$(0, 1), (0, 2), (0, -1), (0, -2), (0, -47) \in \overline{(0, 2)}$.
$(2, 3), (4, 6), (6, 9), (-2, -3), (-4, -6) \in \overline{(2, 3)}$.

5. (Constructing the integers from the natural numbers) Let $S = \mathbb{N} \times \mathbb{N}$. Define the relation $\sim$ on S where $(a, b) \sim (c, d)$ if and only if $a + d = b + c$.

(a) Is $(3, 6) \sim (7, 10)$?
Solution: Yes, because $3 + 10 = 6 + 7$.

(b) Is $(1, 1) \sim (3, 5)$?
Solution: No, because $1 + 5 \neq 1 + 3$.

(c) Prove that $\sim$ is an equivalence relation.

Proof. Reflexive: Let $(a, b) \in \mathbb{N} \times \mathbb{N}$.
Then $a + b = b + a$.
So $(a, b) \sim (a, b)$.
Symmetric: Let $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$.
Suppose $(a, b) \sim (c, d)$.
We know that $a + d = b + c$, because $(a, b) \sim (c, d)$.
So $c + b = d + a$.
So $(c, d) \sim (a, b)$.
Transitive: Let $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$.
Suppose that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.
We know that $a+d = b+c$ and $c+f = d+e$, because $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.
Add these two equations to get $a + c + d + f = b + c + d + e$.
Subtract $c + d$ from both sides to get $a + f = b + e$.
So $(a, b) \sim (e, f)$.

Therefore, $\sim$ is an equivalence relation, because it is reflexive, symmetric, and transitive.

□

(d) List five elements from each of the following equivalence classes: $\overline{(1, 1)}$, $\overline{(1, 2)}$, $\overline{(5, 12)}$.
Solution: Some possible answers:
$(2, 2), (3, 3), (4, 4), (5, 5), (47, 47) \in \overline{(1, 1)}$.
$(2, 3), (3, 4), (4, 5), (5, 6), (47, 48) \in \overline{(1, 2)}$.
$(2, 9), (3, 10), (4, 11), (5, 12), (47, 56) \in \overline{(5, 12)}$.

6. Let $S = \mathbb{Z}$. Define the relation $\sim$ on S where $x \sim y$ if and only if $3x - 5y$ is even. Prove that $\sim$ is an equivalence relation on S.

Proof. Reflexive:

Let $a \in \mathbb{Z}$.

Then $3a - 5a = -2a = 2(-a)$ is even.

Thus, $a \sim a$.

Symmetric:

Let $a, b \in \mathbb{Z}$ and suppose that $a \sim b$.

Then $3a - 5b$ is even and so $3a - 5b = 2k$ for some $k \in \mathbb{Z}$.

Add $8b - 8a$ to both sides to get $3b - 5a = 2k + 8b - 8a$.

Thus $3b - 5a = 2(k + 4b - 4a)$ where $k + 4b - 4a \in \mathbb{Z}$ because $k, a, b \in \mathbb{Z}$.

Thus $3b - 5a$ is even.

So $b \sim a$.

Transitive:

Let $a, b, c \in \mathbb{Z}$ and suppose that $a \sim b$ and $b \sim c$.

Then $3a - 5b$ is even and $3b - 5c$ is even.

So $3a - 5b = 2k_1$ and $3b - 5c = 2k_2$ where $k_1, k_2 \in \mathbb{Z}$.

Adding both equations gives $3a - 2b - 5c = 2k_1 + 2k_2$.

Thus $3a - 5c = 2(k_1 + k_2 + b)$ where $k_1 + k_2 + b \in \mathbb{Z}$ because $k_1, k_2, b \in \mathbb{Z}$.

So $3a - 5c$ is even.

Thus $a \sim c$. $\square$