

Homework 9 Solutions

①

(a) T is linear.

Let $(a,b), (x,y)$ be in $\mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} T((a,b)+(x,y)) &= T(a+x, b+y) = ((a+x)+2(b+y), \cancel{3(a+x)}(b+y)) \\ &= ((a+2b)+(x+2y), (3a-b)+(3x-y)) \\ &= (a+2b, 3a-b) + (x+2y, 3x-y) = T(a,b) + T(x,y) \end{aligned}$$

$$\begin{aligned} \text{and } T(\alpha(a,b)) &= T(\alpha a, \alpha b) = (\alpha a + 2\alpha b, 3\alpha a - \alpha b) \\ &= \alpha(a+2b, 3a-b) = \alpha T(a,b). \end{aligned}$$

(b) T is linear.

Let (a,b,c) and (x,y,z) be in $\mathbb{R}^3$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} T((a,b,c)+(x,y,z)) &= T(a+x, b+y, c+z) \\ &= (2(a+x) - (b+y) + (c+z), (b+y) - 4(c+z)) \\ &= ((2a-b+c) + (2x-y+z), (b-4c) + (y-4z)) \\ &= (2a-b+c, b-4c) + (2x-y+z, y-4z) \\ &= T(a,b,c) + T(x,y,z) \end{aligned}$$

and

$$\begin{aligned} T(\alpha(a,b,c)) &= T(\alpha a, \alpha b, \alpha c) = (2\alpha a - \alpha b + \alpha c, \alpha b - 4\alpha c) \\ &= \alpha(2a-b+c, b-4c) = \alpha T(a,b,c) \end{aligned}$$

(c) T is not linear. For example, let

$\vec{v}_1 = (1, 1)$ and $\vec{v}_2 = (-1, -1)$. Then

$$T(\vec{v}_1 + \vec{v}_2) = T((1, 1) + (-1, -1)) = T(0, 0) = \sqrt{0^2 + 0^2} = 0$$

but

$$\begin{aligned} T(\vec{v}_1) + T(\vec{v}_2) &= T(1, 1) + T(-1, -1) = \cancel{0} \\ &= \sqrt{1^2 + 1^2} + \sqrt{(-1)^2 + (-1)^2} \\ &= \sqrt{2} + \sqrt{2} = 2\sqrt{2}, \end{aligned}$$

So, $T(\vec{v}_1 + \vec{v}_2) \neq T(\vec{v}_1) + T(\vec{v}_2)$.

(d) T is linear. Let $M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $M_2 = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ be in $M_{2,2}$

and $\alpha \in \mathbb{R}$.

Then

$$T(M_1 + M_2) = T\begin{pmatrix} a+x & b+y \\ c+z & d+w \end{pmatrix} = 3(a+x) - 4(b+y) + (c+z) - (d+w)$$

$$\begin{aligned} &= (3a - 4b + c - d) + (3x - 4y + z - w) \\ &= T\begin{pmatrix} a & b \\ c & d \end{pmatrix} + T\begin{pmatrix} x & y \\ z & w \end{pmatrix} = T(M_1) + T(M_2) \end{aligned}$$

and $T(\alpha M_1) = T\begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} = 3(\alpha a) - 4(\alpha b) + (\alpha c) - (\alpha d)$
 $= \alpha[3a - 4b + c - d]$
 $= \alpha T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha T(M_1)$

(e) T is not linear. For example,

$$T\left(\begin{pmatrix} 1 & 1 \\ 5 & 3 \end{pmatrix}\right) + T\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = (1^2 + 1^2) + (1^2 + 0^2) = 3$$

but

$$T\left(\left(\begin{pmatrix} 1 & 1 \\ 5 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\right) = T\left(\begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix}\right) = 2^2 + 1^2 = 5$$

and $5 \neq 3$.

(f) T is linear.

Let $f_1 = a + bx + cx^2$ and $f_2 = d + ex + gx^2$ be in P_2 and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} T(f_1 + f_2) &= T((a+d) + (b+e)x + (c+g)x^2) \\ &= (a+d) + (b+e)(1+x) + (c+g)(1+x)^2 \\ &= [a + b(1+x) + c(1+x)^2] + [d + e(1+x) + g(1+x)^2] \\ &= T(f_1) + T(f_2) \end{aligned}$$

and

$$\begin{aligned} T(\alpha f_1) &= T(\alpha a + \alpha bx + \alpha cx^2) = (\alpha a) + (\alpha b)(1+x) + (\alpha c)(1+x)^2 \\ &= \alpha [a + b(1+x) + c(1+x)^2] \\ &= \alpha T(f_1) \end{aligned}$$

(g) T is not linear.

For example, let $f = 1 + x + x^2$ and $\alpha = 2$.

Then

$$\begin{aligned}\alpha T(f) &= 2T(1+x+x^2) = 2[(1+1) + (1+1)x + (1+1)x^2] \\ &= 2[2 + 2x + 2x^2] = 4 + 4x + 4x^2\end{aligned}$$

but

$$\begin{aligned}T(\alpha f) &= T(2 + 2x + 2x^2) \\ &= (1+2) + (1+2)x + (1+2)x^2 \\ &= 3 + 3x + 3x^2\end{aligned}$$

So, $\alpha T(f) \neq T(\alpha f)$ in this example.

② Let (x, y, z) be in $\mathbb{R}^3$ and

T be linear with $T(1, 0, 0) = (2, 1, -1)$

$T(0, 1, 0) = (0, \pi, \frac{2}{3})$, and $T(0, 0, 1) = (-1, 0, 0)$

Then, since T is linear we have that

$$\begin{aligned}T(x, y, z) &= T(x, 0, 0) + T(0, y, 0) + T(0, 0, z) \\ &= xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1) \\ &= x(2, 1, -1) + y(0, \pi, \frac{2}{3}) + z(-1, 0, 0) \\ &= (2x - z, x + \pi y, -x + \frac{2}{3}y).\end{aligned}$$

③

$$(a) T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\text{So, } [T]_{\beta} = \left([T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)]_{\beta} \mid [T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)]_{\beta} \right) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

(b) We start with $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\text{Then } \vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\text{So, } [\vec{x}]_{\beta} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\text{And } T(\vec{x}) = T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\text{So, } [T(\vec{x})]_{\beta} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

$$\begin{aligned} \begin{pmatrix} -1 \\ -2 \end{pmatrix} &= a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \begin{pmatrix} -1 \\ -2 \end{pmatrix} &= \begin{pmatrix} a+b \\ a+2b \end{pmatrix} \Rightarrow \begin{aligned} 1 &= a+b \\ -2 &= a+2b \end{aligned} \\ &\Downarrow \\ 3 &= -b \\ b &= -3 \\ a &= 4 \end{aligned}$$

And,

$$[T]_{\beta} [\vec{x}]_{\beta} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} = [T(\vec{x})]_{\beta}.$$

Now let $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$$\text{Then } \vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ So, } [\vec{x}]_{\beta} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\text{And } T(\vec{x}) = T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\text{So, } [T(\vec{x})]_{\beta} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}. \text{ And,}$$

$$[T]_{\beta} [\vec{x}]_{\beta} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} = [T(\vec{x})]_{\beta}$$

$$\begin{aligned} \begin{pmatrix} 2 \\ 1 \end{pmatrix} &= a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \begin{aligned} 2 &= a+b \\ 1 &= a+2b \end{aligned} &\Rightarrow \begin{aligned} a &= 3 \\ b &= -1 \end{aligned} \end{aligned}$$

④

(a)

$$\left. \begin{aligned} T(1) &= 1 = 1 \cdot 1 + 0 \cdot x \\ T(x) &= 1 - 2x = 1 \cdot 1 - 2 \cdot x \\ T(x^2) &= 3x = 0 \cdot 1 + 3 \cdot x \end{aligned} \right\} \text{So, } [T]_{\beta}^{\beta'} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 3 \end{pmatrix}$$

(b)

First we look at $\vec{v} = 1 + x + x^2$.

Here, $[\vec{v}]_{\beta} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. And, $T(\vec{v}) = T(1 + x + x^2) = 2 + x$
 So, $[T(\vec{v})]_{\beta'} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

And, $[T]_{\beta}^{\beta'} [\vec{v}]_{\beta} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = [T(\vec{v})]_{\beta'}$

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Now let  $\vec{v} = 2x^2$ .

Then,  $[\vec{v}]_{\beta} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ . And  $T(\vec{v}) = T(2x^2)$   
 $= 0 - (0 - 3 \cdot 2)x$   
 $= 6x = 0 + 6x$   
 So,  $[T(\vec{v})]_{\beta'} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

And,  $[T]_{\beta}^{\beta'} [\vec{v}]_{\beta} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} = [T(\vec{v})]_{\beta'}$

(5)

$$(a) T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$T\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

$$\text{So, } [T]_{\beta} = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}$$

$$(b) \text{ We first let } \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$\text{Then } \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} -1 \\ 0 \end{pmatrix}. \text{ So, } [v]_{\beta} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\text{And } T(\vec{v}) = T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

$$\text{So, } [T(\vec{v})]_{\beta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$\text{And, } [T]_{\beta} [v]_{\beta} = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = [T(\vec{v})]_{\beta}.$$

$$\text{Now let } \vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\text{Then } \vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 0 \end{pmatrix}. \text{ So, } [v]_{\beta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\text{And, } T(\vec{v}) = T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\text{So, } [T(\vec{v})]_{\beta} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$$\text{And, } [T]_{\beta} [\vec{v}]_{\beta} = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-1 \\ 2+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ = [T(\vec{v})]_{\beta}$$

⑥

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a \begin{pmatrix} 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} -3 \\ 4 \end{pmatrix} \Leftrightarrow \begin{cases} 1 = 2a - 3b \\ 0 = a + 4b \end{cases} \Leftrightarrow \begin{cases} b = -1/11 \\ a = 4/11 \end{cases}$$

$$\begin{aligned} (a) \quad T \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{4}{11} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{1}{11} \begin{pmatrix} -3 \\ 4 \end{pmatrix} \\ T \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} -2 \\ -1 \end{pmatrix} = -1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} -3 \\ 4 \end{pmatrix} \end{aligned} \quad \left. \vphantom{\begin{aligned} T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{4}{11} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{1}{11} \begin{pmatrix} -3 \\ 4 \end{pmatrix} } \right\} \text{So, } [T]_{\beta}^{\beta'} = \begin{pmatrix} \frac{4}{11} & -1 \\ -\frac{1}{11} & 0 \end{pmatrix}$$

~~(b)  $T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{4}{11} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{1}{11} \begin{pmatrix} -3 \\ 4 \end{pmatrix}$~~

$$\begin{aligned} (b) \quad I \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{4}{11} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{1}{11} \begin{pmatrix} -3 \\ 4 \end{pmatrix} \\ I \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{11} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{2}{11} \begin{pmatrix} -3 \\ 4 \end{pmatrix} \end{aligned} \quad \left. \vphantom{\begin{aligned} I \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{4}{11} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{1}{11} \begin{pmatrix} -3 \\ 4 \end{pmatrix} } \right\} \text{So, } [I]_{\beta}^{\beta'} = \begin{pmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{pmatrix}, \quad Q =$$

$$\begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a \begin{pmatrix} 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} -3 \\ 4 \end{pmatrix} \\ 0 = 2a - 3b \\ 1 = a + 4b \end{cases} \Rightarrow \begin{cases} b = 2/11 \\ a = 3/11 \end{cases}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = a \begin{pmatrix} 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} -3 \\ 4 \end{pmatrix} \Leftrightarrow \begin{cases} 1 = 2a - 3b \\ 1 = a + 4b \end{cases} \Leftrightarrow \begin{cases} b = 1/11 \\ a = 7/11 \end{cases}$$

(c) Let  $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

Then  $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{7}{11} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{11} \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ . So,  $[\vec{v}]_{\beta'} = \begin{pmatrix} 7/11 \\ 1/11 \end{pmatrix}$

And  $[\vec{v}]_{\beta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

And,  $Q[\vec{v}]_{\beta} = Q \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7/11 \\ 1/11 \end{pmatrix} = [\vec{v}]_{\beta'}$

Now let  $\vec{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ .

Then  $\vec{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} = 1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ . So,  $[\vec{v}]_{\beta'} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

And  $[\vec{v}]_{\beta} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ .

$$\begin{cases} \begin{pmatrix} -1 \\ 5 \end{pmatrix} = a \begin{pmatrix} 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} -3 \\ 4 \end{pmatrix} \\ -1 = 2a - 3b \\ 5 = a + 4b \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 1 \end{cases}$$

And,  $Q[\vec{v}]_{\beta} = \begin{pmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{pmatrix} \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = [\vec{v}]_{\beta'}$



(d) From b we have that  $Q = \begin{pmatrix} 4/11 & 3/11 \\ -1/11 & 2/11 \end{pmatrix}$ .

$$\text{So, } Q^{-1} = \frac{1}{\frac{4}{11} \cdot \frac{2}{11} - \frac{3}{11}(-\frac{1}{11})} \begin{pmatrix} 2/11 & -3/11 \\ 1/11 & 4/11 \end{pmatrix} = \frac{1}{(\frac{11}{11^2})} \begin{pmatrix} 2/11 & -3/11 \\ 1/11 & 4/11 \end{pmatrix} \\ = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$$

And

$$\left. \begin{aligned} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} -2 \\ -1 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \right\} \text{So, } [T]_{\beta} = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix},$$

And

$$\left. \begin{aligned} T\begin{pmatrix} 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ T\begin{pmatrix} -3 \\ 4 \end{pmatrix} &= \begin{pmatrix} -11 \\ -4 \end{pmatrix} \end{aligned} \right\} \text{So, } [T]_{\beta'} = \begin{pmatrix} -\frac{3}{11} & -\frac{56}{11} \\ -\frac{2}{11} & \frac{3}{11} \end{pmatrix},$$

$$\left( \begin{aligned} \begin{pmatrix} -11 \\ -4 \end{pmatrix} &= a \begin{pmatrix} 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} -3 \\ 4 \end{pmatrix} \\ -11 &= 2a - 3b \\ -4 &= a + 4b \end{aligned} \right\} \begin{aligned} b &= \frac{3}{11} \\ a &= -\frac{56}{11} \end{aligned}$$

Here we go!

$$Q^{-1}[T]_{\beta'}Q = Q^{-1} \begin{pmatrix} -3/11 & -56/11 \\ -2/11 & 3/11 \end{pmatrix} \begin{pmatrix} 4/11 & 3/11 \\ -1/11 & 2/11 \end{pmatrix} = Q^{-1} \begin{pmatrix} 4/11 & -1 \\ -1/11 & 0 \end{pmatrix} \\ = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 4/11 & -1 \\ -1/11 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} = [T]_{\beta}$$

7

$$(a) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So,  $[I]_{\beta}^{\beta'} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{cases} 2 = a - b \\ 1 = a + 3b \end{cases} \Rightarrow \begin{cases} b = -1/4 \\ a = 7/4 \end{cases}$$

$$(b) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \frac{7}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix} + d \begin{pmatrix} -1 \\ 3 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\begin{cases} d = 1/2 \\ c = -1/2 \end{cases} \begin{cases} -1 = c - d \\ 1 = c + 3d \end{cases}$$

So,  ~~$[I]_{\beta}^{\beta'}$~~   $[I]_{\beta}^{\beta'} = \begin{pmatrix} 7/4 & -1/2 \\ -1/4 & 1/2 \end{pmatrix}$

⑧ (a)

$$\vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \text{ So, } [\vec{v}]_{\beta} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

$$\text{And, } \vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \text{ So, } [\vec{v}]_{\beta'} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{And, } Q[\vec{v}]_{\beta} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = [\vec{v}]_{\beta'}.$$

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$$\begin{cases} -1 = 2a - b \\ 2 = a + b \end{cases} \quad \begin{cases} b = 5/3 \\ a = 1/3 \end{cases}$$

(b) $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{5}{3} \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \text{ So, } [\vec{v}]_{\beta} = \begin{pmatrix} 1/3 \\ 5/3 \end{pmatrix}.$

And

$$\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -1/4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3/4 \begin{pmatrix} -1 \\ 3 \end{pmatrix}. \text{ So, } [\vec{v}]_{\beta'} = \begin{pmatrix} -1/4 \\ 3/4 \end{pmatrix}.$$

$$\begin{cases} -1 = a + (-b) \\ 2 = a + 3b \end{cases} \quad \begin{cases} b = 3/4 \\ a = -1/4 \end{cases}$$

And,

$$Q[\vec{v}]_{\beta} = \begin{pmatrix} 7/4 & -1/2 \\ -1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 1/3 \\ 5/3 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 3/4 \end{pmatrix} = [\vec{v}]_{\beta'}.$$