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HW 7

Part 1

Solutions

① (a)

$$\text{span}\left(\{\langle 0,1\rangle, \langle 1,1\rangle, \langle -3,2\rangle\}\right)$$

$$= \left\{ c_1\langle 0,1\rangle + c_2\langle 1,1\rangle + c_3\langle -3,2\rangle \;\middle|\; \begin{array}{l} c_1, c_2, c_3 \\ \text{are real} \\ \text{numbers} \end{array} \right\}$$

Five example vectors in the above span:

$$1\cdot\langle 0,1\rangle + 1\cdot\langle 1,1\rangle + 1\cdot\langle -3,2\rangle = \boxed{\langle -2,4\rangle}$$

$$0\cdot\langle 0,1\rangle + 0\cdot\langle 1,1\rangle + 0\cdot\langle -3,2\rangle = \boxed{\langle 0,0\rangle}$$

$$10\cdot\langle 0,1\rangle + \pi\langle 1,1\rangle + 0\cdot\langle -3,2\rangle = \boxed{\langle \pi, \pi+10\rangle}$$

$$0\cdot\langle 0,1\rangle + 1\cdot\langle 1,1\rangle - 5\cdot\langle -3,2\rangle = \boxed{\langle 16,-9\rangle}$$

$$2\cdot\langle 0,1\rangle - 1\cdot\langle 1,1\rangle + 1\cdot\langle -3,2\rangle = \boxed{\langle -4,3\rangle}$$

1 (b)

$$\text{span}\left(\{\langle 0,-2,2\rangle, \langle 1,3,-1\rangle\}\right)$$

$$= \left\{ c_1\langle 0,-2,2\rangle + c_2\langle 1,3,-1\rangle \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}$$

<u>Five example vectors in the above span:</u>

$$0\cdot\langle 0,-2,2\rangle + 0\cdot\langle 1,3,-1\rangle = \boxed{\langle 0,0,0\rangle}$$

$$1\cdot\langle 0,-2,2\rangle + 0\cdot\langle 1,3,-1\rangle = \boxed{\langle 0,-2,2\rangle}$$

$$0\cdot\langle 0,-2,2\rangle + 1\cdot\langle 1,3,-1\rangle = \boxed{\langle 1,3,-1\rangle}$$

$$\frac{1}{2}\cdot\langle 0,-2,2\rangle - 2\langle 1,3,-1\rangle = \boxed{\langle -2,-7,3\rangle}$$

$$5\cdot\langle 0,-2,2\rangle + 1\cdot\langle 1,3,-1\rangle = \boxed{\langle 1,-7,9\rangle}$$

1(c)

$$\operatorname{span}(\{2, 1+x\})$$

$$= \{ c_1 \cdot 2 + c_2 \cdot (1+x) \mid c_1, c_2 \text{ are real numbers} \}$$

Five example vectors in the above span:

$$0 \cdot 2 + 0 \cdot (1+x) = \boxed{0}$$

$$1 \cdot 2 + 0 \cdot (1+x) = \boxed{2}$$

$$0 \cdot 2 + \frac{1}{2}(1+x) = \boxed{\frac{1}{2} + \frac{1}{2}x}$$

$$\left(-\frac{1}{2}\right) \cdot 2 + \pi \cdot (1+x) = \boxed{(-1+\pi) + \pi x}$$

$$10 \cdot 2 - 10 \cdot (1+x) = \boxed{10 - 10x}$$

① (d)

$$\operatorname{span}\left(\{-1-2x,\ x^2,\ 1+x+x^2\}\right)$$
$$= \left\{ c_1(-1-2x) + c_2(x^2) + c_3((1+x+x^2)) \;\middle|\; c_1, c_2, c_3 \in \mathbb{R} \right\}$$

<u>Five example vectors in the above span:</u>

$$1\cdot(-1-2x) + 0\cdot x^2 + 0\cdot(1+x+x^2) = \boxed{-1-2x}$$

$$-1\cdot(-1-2x) + 1\cdot x^2 - 1\cdot(1+x+x^2) = \boxed{x}$$

$$2\cdot(-1-2x) - 2\cdot x^2 + 1\cdot(1+x+x^2) = \boxed{-1-3x-x^2}$$

$$4\cdot(-1-2x) + 0\cdot x^2 + 0\cdot(1+x+x^2) = \boxed{-4-8x}$$

$$0\cdot(-1-2x) + 5\cdot x^2 - 5\cdot(1+x+x^2) = \boxed{-5-5x}$$

② (a)

We want to know if we can write

$$\langle 2,2,2\rangle = c_1\langle 0,-2,2\rangle + c_2\langle 1,3,-1\rangle$$

Let's see.

The above equation becomes:

$$\langle 2,2,2\rangle = \langle c_2, -2c_1+3c_2, 2c_1-c_2\rangle$$

Now equate each component to get

$$\boxed{\begin{aligned} 2 &= c_2 \\ 2 &= -2c_1+3c_2 \\ 2 &= 2c_1-c_2 \end{aligned}}$$

Let's see if we can solve this system

$$\left(\begin{array}{cc|c} 0 & 1 & 2 \\ -2 & 3 & 2 \\ 2 & -1 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{cc|c} 2 & -1 & 2 \\ -2 & 3 & 2 \\ 0 & 1 & 2 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 1 \\ -2 & 3 & 2 \\ 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{2R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 1 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cc|c} 1 & -1/2 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array}\right) \longrightarrow$$

$$\xrightarrow{-2R_2+R_3\to R_3} \left(\begin{array}{cc|c} 1 & -1/2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

So we have

$$\boxed{\begin{aligned} c_1 - \tfrac{1}{2}c_2 &= 1 \quad ① \\ c_2 &= 2 \quad ② \\ 0 &= 0 \end{aligned}}$$

② gives $c_2 = 2$

① gives $c_1 = 1 + \frac{1}{2}c_2$

$= 1 + \frac{1}{2}(2) = 2$

Thus,

$$\langle 2,2,2\rangle = 2\cdot\langle 0,-2,2\rangle + 2\cdot\langle 1,3,-1\rangle$$

$$= 2\cdot\vec{u} + 2\vec{v}$$

So,

$\langle 2,2,2\rangle$ is in the span of $\vec{u}$ and $\vec{v}$.

② (b)

We want to know if we can write

$$\langle 3,1,5\rangle = c_1\langle 0,-2,2\rangle + c_2\langle 1,3,-1\rangle$$

Let's see.

The above equation becomes:

$$\langle 3,1,5\rangle = \langle c_2, -2c_1+3c_2, 2c_1-c_2\rangle$$

Now equate each component to get

$$\boxed{\begin{aligned} 3 &= c_2 \\ 1 &= -2c_1+3c_2 \\ 5 &= 2c_1-c_2 \end{aligned}}$$

Let's see if we can solve this system

$$\left(\begin{array}{cc|c} 0 & 1 & 3 \\ -2 & 3 & 1 \\ 2 & -1 & 5 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{cc|c} 2 & -1 & 5 \\ -2 & 3 & 1 \\ 0 & 1 & 3 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{5}{2} \\ -2 & 3 & 1 \\ 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{2R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 5/2 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cc|c} 1 & -1/2 & 5/2 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{array}\right) \longrightarrow$$

$$\xrightarrow{-2R_2+R_3 \to R_3} \left(\begin{array}{cc|c} 1 & -1/2 & 5/2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

So we have

$$\boxed{\begin{aligned} c_1 - \tfrac{1}{2} c_2 &= 5/2 \quad (1) \\ c_2 &= 3 \quad (2) \\ 0 &= 0 \end{aligned}}$$

(2) gives $c_2 = 3$

(1) gives $c_1 = 5/2 + \frac{1}{2} c_2$

$= 5/2 + \frac{1}{2}(3)$

$= 8/2 = 4$

Thus,

$$\langle 2,2,2 \rangle = 4 \cdot \langle 0,-2,2 \rangle + 3 \cdot \langle 1,3,-1 \rangle$$

$$= 4 \vec{u} + 3 \vec{v}$$

So,

$\langle 2,2,2 \rangle$ is in the span of $\vec{u}$ and $\vec{v}$.

5/2

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② (c)

We want to know if we can write

$$\langle 0,4,5\rangle = c_1\langle 0,-2,2\rangle + c_2\langle 1,3,-1\rangle$$

Let's see.

The above equation becomes:

$$\langle 0,4,5\rangle = \langle c_2, -2c_1+3c_2, 2c_1-c_2\rangle$$

Now equate each component to get

$$\boxed{\begin{aligned} 0 &= c_2 \\ 4 &= -2c_1+3c_2 \\ 5 &= 2c_1-c_2 \end{aligned}}$$

$\Leftarrow$ Let's see if we can solve this system

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 4 \\ 2 & -1 & 5 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{cc|c} 2 & -1 & 5 \\ -2 & 3 & 4 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 5/2 \\ -2 & 3 & 4 \\ 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{2R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 5/2 \\ 0 & 2 & 9 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cc|c} 1 & -1/2 & 5/2 \\ 0 & 1 & 0 \\ 0 & 2 & 9 \end{array}\right) \longrightarrow$$

$$\xrightarrow{-2R_2+R_3 \to R_3} \left(\begin{array}{cc|c} 1 & -1/2 & 5/2 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{array}\right)$$

So we have

$$\boxed{\begin{aligned} c_1 - \tfrac{1}{2} c_2 &= 5/2 \quad (1) \\ c_2 &= 0 \quad (2) \\ 0 &= 9 \quad (3) \end{aligned}}$$

This last equation $0=9$ shows that the system has no solutions

Thus, there is no solution to

$$\langle 0,4,5\rangle = c_1 \cdot \langle 0,-2,2\rangle + c_2 \cdot \langle 1,3,-1\rangle$$
$$= c_1 \vec{u} + c_2 \vec{v}$$

$\langle 0,4,5\rangle$ is <u>not</u> in the span of $\vec{u}$ and $\vec{v}$.

2 (d) You can proceed as in the previous problems, but this one is easy to solve.

We have

$$\langle 0,0,0\rangle = 0\cdot\langle 0,-2,2\rangle + 0\cdot\langle 1,3,-1\rangle$$
$$= 0\cdot\vec{u} + 0\cdot\vec{v}$$

Thus, $\langle 0,0,0\rangle$ is in the span of $\vec{u}$ and $\vec{v}$.

③ (a)

We want to know if we can write

$$3+2x+x^2+2x^3 = c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3$$
$$= c_1(2+x+4x^2) + c_2(1-x+3x^2) + c_3(1+x^3)$$

This simplifies to

$$3+2x+x^2+2x^3 = (2c_1+c_2+c_3) + (c_1-c_2)x + (4c_1+3c_2)x^2 + c_3x^3$$

Now equate the coefficients of both sides to get:

$$\boxed{\begin{aligned} 3 &= 2c_1 + c_2 + c_3 \\ 2 &= c_1 - c_2 \\ 1 &= 4c_1 + 3c_2 \\ 2 &= c_3 \end{aligned}}$$

Now we must see if this system has a solution or not

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 1 & -1 & 0 & 2 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & 3 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow[-4R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 3 & 1 & -1 \\ 0 & 7 & 0 & -7 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 7 & 0 & -7 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{\frac{1}{7}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{-3R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{-R_3 + R_4 \to R_4} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow$$

The reduced system is

$$\boxed{\begin{aligned} c_1 - c_2 \quad &= 2 \\ c_2 \quad &= -1 \\ c_3 &= 2 \\ 0 &= 0 \end{aligned}} \quad \begin{matrix} 1 \\ 2 \\ 3 \\ \end{matrix}$$

We get the solution

$$\boxed{\begin{aligned} &c_3 = 2 \\ &c_2 = -1 \\ &c_1 = 2 + c_2 = 2 - 1 = 1 \end{aligned}}$$

Thus,

$$3 + 2x + x^2 + 2x^3 = 1 \cdot \vec{p}_1 - 1 \cdot \vec{p}_2 + 2 \cdot \vec{p}_3$$

So,

$3 + 2x + x^2 + 2x^3$ is in the span of $\vec{p}_1, \vec{p}_2, \vec{p}_3$

③ (b)

We want to know if we can write

$$1 + x = c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3$$

$$1 + 1\cdot x + 0\cdot x^2 + 0 x^3 = c_1(2 + x + 4x^2) + c_2(1 - x + 3x^2) + c_3(1 + x^3)$$

This simplifies to

$$1 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3 = (2c_1 + c_2 + c_3) + (c_1 - c_2)x + (4c_1 + 3c_2)x^2 + c_3 x^3$$

Now equate the coefficients of both sides to get:

$$\boxed{\begin{aligned} 1 &= 2c_1 + c_2 + c_3 \\ 1 &= c_1 - c_2 \\ 0 &= 4c_1 + 3c_2 \\ 0 &= c_3 \end{aligned}}$$

Now we must see if this system has a solution or not

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[-4R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 3 & 1 & -1 \\ 0 & 7 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 7 & 0 & -4 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\frac{1}{7}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -4/7 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-3R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -4/7 \\ 0 & 0 & 1 & 5/7 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-R_3 + R_4 \to R_4} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -4/7 \\ 0 & 0 & 1 & 5/7 \\ 0 & 0 & 0 & -5/7 \end{array}\right)$$

The reduced system is

$$\boxed{\begin{array}{rrrrl} c_1 & - c_2 & & = & 1 \\ & c_2 & & = & -4/7 \\ & & c_3 & = & 5/7 \\ & & 0 & = & -5/7 \end{array}} \quad \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Equation 4 is $0 = -5/7$.
Which isn't true.
Hence the system has no solutions.

Thus,

$$3 + 2x + x^2 + 2x^3 = c_1 \cdot \vec{P}_1 \; c_2 \cdot \vec{P}_2 + c_3 \cdot \vec{P}_3$$

Cannot be solved for c_1, c_2, c_3.

So,

$3 + 2x + x^2 + 2x^3$ is <u>not</u> in the span of $\vec{P}_1, \vec{P}_2, \vec{P}_3$

3 (c)

We have that

$$0 = 0 \cdot \vec{p}_1 + 0 \cdot \vec{p}_2 + 0\,\vec{p}_3$$

Thus, 0 is in the span of

$\vec{p}_1, \vec{p}_2, \vec{p}_3$.

③ (d)

We want to know if we can write

$$4 - x + 10x^2 = c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3$$

$$4 - x + 10x^2 = c_1(2 + x + 4x^2) + c_2(1 - x + 3x^2) + c_3(1 + x^3)$$

This simplifies to

$$4 - x + 10x^2 + 0x^3 = (2c_1 + c_2 + c_3) + (c_1 - c_2)x + (4c_1 + 3c_2)x^2 + c_3 x^3$$

Now equate the coefficients of both sides to get:

$$\boxed{\begin{aligned} 4 &= 2c_1 + c_2 + c_3 \\ -1 &= c_1 - c_2 \\ 10 &= 4c_1 + 3c_2 \\ 0 &= c_3 \end{aligned}}$$

Now we must see if this system has a solution or not

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 1 & -1 & 0 & -1 \\ 4 & 3 & 0 & 10 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 2 & 1 & 1 & 4 \\ 4 & 3 & 0 & 10 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[-4R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 3 & 1 & 6 \\ 0 & 7 & 0 & 14 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 7 & 0 & 14 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\frac{1}{7}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-3R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-R_3 + R_4 \to R_4} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow$$

The reduced system is

$$\boxed{\begin{array}{rcl} c_1 - c_2 & = & -1 \\ c_2 & = & 2 \\ c_3 & = & 0 \\ 0 & = & 0 \end{array}} \quad \begin{array}{c} 1 \\ 2 \\ 3 \\ \\ \end{array}$$

We get the solution

$$\boxed{\begin{array}{l} c_3 = 0 \\ c_2 = 2 \\ c_1 = -1 + c_2 = -1 + 2 = 1 \end{array}}$$

Thus,

$$4 - x + 10x^2 = 1 \cdot \vec{p}_1 + 2 \cdot \vec{p}_2 + 0 \cdot \vec{p}_3$$

So,

$4 - x + 10x^2$ is in the span of $\vec{p}_1, \vec{p}_2, \vec{p}_3$

④ (a)

$\boxed{\text{Method 1}}$

We want to know if no matter what $\vec{b}$ is, can we always solve the equation

$$\vec{b} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

for c_1, c_2, c_3 ?

Let $\vec{b} = \langle b_1, b_2, b_3 \rangle$,

We want to see if we can always solve

$$\langle b_1, b_2, b_3 \rangle = c_1 \langle 2,2,2 \rangle + c_2 \langle 4,1,2 \rangle + c_3 \langle 0,1,1 \rangle$$

This equation becomes

$$\langle b_1, b_2, b_3 \rangle = \langle 2c_1 + 4c_2, 2c_1 + c_2 + c_3, 2c_1 + 2c_2 + c_3 \rangle$$

which is equivalent to the system

$$\boxed{\begin{aligned} b_1 &= 2c_1 + 4c_2 \\ b_2 &= 2c_1 + c_2 + c_3 \\ b_3 &= 2c_1 + 2c_2 + c_3 \end{aligned}}$$

Let's see if we can solve this system.

$$\left(\begin{array}{ccc|c} 2 & 4 & 0 & b_1 \\ 2 & 1 & 1 & b_2 \\ 2 & 2 & 1 & b_3 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 2 & 0 & b_1/2 \\ 2 & 1 & 1 & b_2 \\ 2 & 2 & 1 & b_3 \end{array}\right)$$

$$\xrightarrow[-2R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & b_1/2 \\ 0 & -3 & 1 & b_2-b_1 \\ 0 & -2 & 1 & b_3-b_1 \end{array}\right)$$

$$\xrightarrow{\frac{-1}{3}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & b_1/2 \\ 0 & 1 & -1/3 & \frac{1}{3}b_1 - \frac{1}{3}b_2 \\ 0 & -2 & 1 & -b_1 + b_3 \end{array}\right)$$

$$\xrightarrow{2R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & b_1/2 \\ 0 & 1 & -1/3 & \frac{1}{3}b_1 - \frac{1}{3}b_2 \\ 0 & 0 & 1/3 & -\frac{1}{3}b_1 - \frac{2}{3}b_2 + b_3 \end{array}\right)$$

$$\xrightarrow{3R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & b_1/2 \\ 0 & 1 & -1/3 & \frac{1}{3}b_1 - \frac{1}{3}b_2 \\ 0 & 0 & 1 & -b_1 - 2b_2 + 3b_3 \end{array}\right)$$

The reduced system is:

$$\boxed{\begin{aligned} c_1 + 2c_2 \quad &= \frac{b_1}{2} \\ c_2 - \frac{1}{3}c_3 &= \frac{b_1}{3} - \frac{b_2}{3} \\ c_3 &= -b_1 - 2b_2 + 3b_3 \end{aligned}} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

③ gives $c_3 = -b_1 - 2b_2 + 3b_3$

② gives $c_2 = \frac{1}{3}b_1 - \frac{1}{3}b_2 + \frac{1}{3}c_3 = \frac{1}{3}b_1 - \frac{1}{3}b_2 - \frac{1}{3}b_1 - \frac{2}{3}b_2 + b_3$

$$= -b_2 + b_3$$

① gives $c_1 = \frac{1}{2}b_1 - 2c_2 = \frac{1}{2}b_1 - 2(-b_2 + b_3)$

$$= \frac{1}{2}b_1 + 2b_2 - 2b_3$$

Therefore, no matter what $\langle b_1, b_2, b_3 \rangle$ is we have

$$\langle b_1, b_2, b_3 \rangle = \left(\tfrac{1}{2}b_1 + 2b_2 - 2b_3\right)\langle 2, 2, 2 \rangle$$
$$+ \left(-b_2 + b_3\right)\langle 4, 1, 2 \rangle$$
$$+ \left(-b_1 - 2b_2 + 3b_3\right)\langle 0, 1, 1 \rangle$$

Thus, every vector $\langle b_1, b_2, b_3 \rangle$ in $\mathbb{R}^3$ is in the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

So, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ span $\mathbb{R}^3$.

$\boxed{\text{Method 2}}$

Since we have three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in a three-dimensional space $\mathbb{R}^3$, if we show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent, then they must span $\mathbb{R}^3$.

Suppose,

$$c_1\langle 2,2,2\rangle + c_2\langle 4,1,2\rangle + c_3\langle 0,1,1\rangle = \langle 0,0,0\rangle$$

Then,

$$\langle 2c_1+4c_2,\ 2c_1+c_2+c_3,\ 2c_1+2c_2+c_3\rangle = \langle 0,0,0\rangle$$

which becomes

$$\boxed{\begin{aligned} 2c_1+4c_2 &= 0 \\ 2c_1+c_2+c_3 &= 0 \\ 2c_1+2c_2+c_3 &= 0 \end{aligned}}$$

Let's try to solve this system

$$\left(\begin{array}{ccc|c} 2 & 4 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[-2R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{2R_2+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 1/3 & 0 \end{array}\right) \xrightarrow{3R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The reduced system is

$$\boxed{\begin{aligned} c_1 + 2c_2 \quad\quad &= 0 \\ c_2 - \tfrac{1}{3}c_3 &= 0 \\ c_3 &= 0 \end{aligned}}$$

which gives $c_3 = 0$

$c_2 = \frac{1}{3}c_3 = \frac{1}{3}\cdot 0 = 0$

$c_1 = -2c_2 = -2\cdot 0 = 0.$

Therefore, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are lin. ind. and thus span all of $\mathbb{R}^3$.

(4) (b) Same method as in 4(a).

Can we always solve

$\langle b_1, b_2, b_3 \rangle = c_1 \langle 2,-1,3 \rangle + c_2 \langle 4,1,2 \rangle + c_3 \langle 8,-1,8 \rangle$

for c_1, c_2, c_3 ?

The above becomes

$\langle b_1, b_2, b_3 \rangle = \langle 2c_1 + 4c_2 + 8c_3, -c_1 + c_2 - c_3, 3c_1 + 2c_2 + 8c_3 \rangle$

which is equivalent to

$$\boxed{\begin{aligned} b_1 &= 2c_1 + 4c_2 + 8c_3 \\ b_2 &= -c_1 + c_2 - c_3 \\ b_3 &= 3c_1 + 2c_2 + 8c_3 \end{aligned}}$$

Can we solve this system for all b_1, b_2, b_3 ?

$$\left(\begin{array}{ccc|c} 2 & 4 & 8 & b_1 \\ -1 & 1 & -1 & b_2 \\ 3 & 2 & 8 & b_3 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & b_1/2 \\ -1 & 1 & -1 & b_2 \\ 3 & 2 & 8 & b_3 \end{array}\right)$$

$$\xrightarrow[-3R_1 + R_3 \to R_3]{R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & b_1/2 \\ 0 & 3 & 3 & b_2 + b_1/2 \\ 0 & -4 & -4 & b_3 - 3b_1/2 \end{array}\right) \longrightarrow$$

$$\xrightarrow{\frac{1}{3}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & b_1/2 \\ 0 & 1 & 1 & b_2/3 + b_1/6 \\ 0 & -4 & -4 & b_3 - 3b_1/2 \end{array}\right)$$

$$\xrightarrow{4R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & b_1/2 \\ 0 & 1 & 1 & b_2/3 + b_1/6 \\ 0 & 0 & 0 & -\frac{b_1}{3} + \frac{4}{3}b_2 + b_3 \end{array}\right)$$

The reduced system is:

$$\boxed{\begin{aligned} c_1 + 2c_2 + 4c_3 &= \frac{b_1}{2} && (1) \\ c_2 + c_3 &= \frac{b_2}{3} + \frac{b_1}{6} && (2) \\ 0 &= -\frac{b_1}{3} + \frac{4}{3}b_2 + b_3 && (3) \end{aligned}}$$

Note that equation (3) cannot be solved for every $\langle b_1, b_2, b_3 \rangle$. For example, if $\langle b_1, b_2, b_3 \rangle = \langle 1, 0, 0 \rangle$ then equation (3) becomes $0 = -\frac{1}{3}$ which isn't solvable.

Hence not every $\langle b_1, b_2, b_3 \rangle$ is in the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Thus $\vec{v}_1, \vec{v}_2, \vec{v}_3$ do <u>not</u> span all of $\mathbb{R}^3$.

(5) (a) Since the vectors in 4(a) span $\mathbb{R}^3$ and there are 3 vectors and the dimension of $\mathbb{R}^3$ is 3, by a theorem from class they must be linearly independent also and hence a basis for $\mathbb{R}^3$.

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(5) (b) Since the vectors in 4(b) do not span $\mathbb{R}^3$, they are not a basis for $\mathbb{R}^3$

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⑥ (a) $\boxed{\text{Method 1}}$

Consider the equation $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{0}$.
We can always find the solution $c_1 = c_2 = c_3 = 0$ to this equation. If this is the <u>only</u> solution then $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are linearly independent. If there are more solutions then $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are lin. dependent.
Let's see what happens.

The equation $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{0}$ is:

$$c_1\langle 3,-1\rangle + c_2\langle 4,5\rangle + c_3\langle -4,7\rangle = \langle 0,0\rangle$$

This is equivalent to

$$\langle 3c_1 + 4c_2 - 4c_3, -c_1 + 5c_2 + 7c_3\rangle = \langle 0,0\rangle$$

Thus,

$$\boxed{\begin{aligned} 3c_1 + 4c_2 - 4c_3 &= 0 \\ -c_1 + 5c_2 + 7c_3 &= 0 \end{aligned}}$$

Let's solve this system.

$$\left(\begin{array}{ccc|c} 3 & 4 & -4 & 0 \\ -1 & 5 & 7 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} -1 & 5 & 7 & 0 \\ 3 & 4 & -4 & 0 \end{array}\right) \longrightarrow$$

$$\xrightarrow{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & -5 & -7 & 0 \\ 3 & 4 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{-3R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -5 & -7 & 0 \\ 0 & 19 & 17 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{19}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -5 & -7 & 0 \\ 0 & 1 & \frac{17}{19} & 0 \end{array}\right)$$

So, we get:

$$\boxed{\begin{aligned} c_1 - 5c_2 - 7c_3 &= 0 \\ c_2 + \tfrac{17}{19}c_3 &= 0 \end{aligned}}$$

Leading variables: c_1, c_2

Free variables: c_3

$$\Downarrow$$

$$\boxed{\begin{aligned} c_1 &= 5c_2 + 7c_3 \\ c_2 &= \tfrac{-17}{19}c_3 \end{aligned}} \Rightarrow \boxed{\begin{aligned} c_3 &= t \\ c_2 &= \tfrac{-17}{19}t \\ c_1 &= 5c_2 + 7c_3 \\ &= -\tfrac{85}{19}t + 7t \\ &= \tfrac{48}{19}t \end{aligned}} \rightarrow$$

So,

$$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{0}$$

can be solved by

$$\left(\frac{48}{19}t\right)\cdot\vec{u}_1 - \left(\frac{17}{19}t\right)\cdot\vec{u}_2 + t\,\vec{u}_3 = \vec{0}$$

for any t.

In particular, say we set $t=19$,

Then we get:

$$48\vec{u}_1 - 17\vec{u}_2 + 19\vec{u}_3 = \vec{0}.$$

Thus, $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are linearly **dependent.**

$\boxed{\text{Method 2}}$: The dimension of $\mathbb{R}^2$ is 2.

Thus, if we have more than 2 vectors in $\mathbb{R}^2$ they must be linearly dependent by a theorem in class. Since we have 3 vectors in a 2-dimensional space, $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are lin. dep.

⑥ (b)

Consider the equation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}.$$

If the only solution to this equation is $c_1 = c_2 = c_3 = 0$, then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent. If there are more solutions then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent. Let's see what happens.

The above equation becomes

$$c_1 \langle -3, 0, 4 \rangle + c_2 \langle 5, -1, 2 \rangle + c_3 \langle 1, 1, 3 \rangle = \langle 0, 0, 0 \rangle$$

which becomes

$$\langle -3c_1 + 5c_2 + c_3, -c_2 + c_3, 4c_1 + 2c_2 + 3c_3 \rangle = \langle 0, 0, 0 \rangle$$

which is equivalent to

$$\boxed{\begin{aligned} -3c_1 + 5c_2 + c_3 &= 0 \\ -c_2 + c_3 &= 0 \\ 4c_1 + 2c_2 + 3c_3 &= 0 \end{aligned}}$$

Let's solve this system.

$$\left(\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right) \xrightarrow{-\frac{1}{3}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & -\frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{-4R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -5/3 & -1/3 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & \frac{26}{3} & \frac{13}{3} & 0 \end{array}\right)$$

$$\xrightarrow[3R_3 \to R_3]{-R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -5/3 & -1/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 26 & 13 & 0 \end{array}\right)$$

$$\xrightarrow{-26R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -5/3 & -1/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 39 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{39}R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -5/3 & -1/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

This becomes

$$\boxed{\begin{aligned} c_1 - \tfrac{5}{3}c_2 - \tfrac{1}{3}c_3 &= 0 \quad (1) \\ c_2 - c_3 &= 0 \quad (2) \\ c_3 &= 0 \quad (3) \end{aligned}}$$

(3) gives $c_3 = 0$

(2) gives $c_2 = c_3 = 0$

(1) gives $c_1 = \frac{5}{3}c_2 + \frac{1}{3}c_3 = \frac{5}{3}\cdot 0 + \frac{1}{3}\cdot 0 = 0$

Thus, the only solution to

$$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{0}$$

is $c_1 = c_2 = c_3 = 0$.

So, $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are linearly independent.

⑥ (c)

Consider the equation

$$c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3 = \vec{0}.$$

If the only solution to this equation is $c_1 = c_2 = c_3 = 0$, then $\vec{p}_1, \vec{p}_2, \vec{p}_3$ are linearly independent. If there are more solutions then $\vec{p}_1, \vec{p}_2, \vec{p}_3$ are linearly dependent. Let's see what happens.

The above equation becomes

$$c_1(3-2x+x^2) + c_2(1+x+x^2) + c_3(6-4x+2x^2) = \underbrace{0+0x+0x^2}_{\vec{0}}$$

Grouping like terms gives

$$(3c_1 + c_2 + 6c_3) + (-2c_1 + c_2 - 4c_3)x + (c_1 + c_2 + 2c_3) = 0 + 0x + 0x^2$$

Equating coefficients gives

$$\boxed{\begin{aligned} 3c_1 + c_2 + 6c_3 &= 0 \\ -2c_1 + c_2 - 4c_3 &= 0 \\ c_1 + c_2 + 2c_3 &= 0 \end{aligned}}$$

$\leftarrow$ Let's solve this system.

$$\left(\begin{array}{ccc|c} 3 & 1 & 6 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -2 & 1 & -4 & 0 \\ 3 & 1 & 6 & 0 \end{array}\right)$$

$$\xrightarrow[-3R_1 + R_3 \to R_3]{2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[\frac{-1}{2}R_3 \to R_3]{\frac{1}{3}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The reduced system is:

$$\boxed{\begin{aligned} c_1 + c_2 + 2c_3 &= 0 \\ c_2 &= 0 \\ 0 &= 0 \end{aligned}}$$

Leading variables are c_1, c_2.

Free variable is c_3

$$\boxed{\begin{aligned} c_1 &= -c_2 - 2c_3 \quad (1) \\ c_2 &= 0 \quad (2) \end{aligned}}$$

$\Rightarrow$

$$\boxed{\begin{aligned} c_3 &= t \quad \text{(set free variable)} \\ (2)\ c_2 &= 0 \\ (1)\ c_1 &= -c_2 - 2c_3 \\ &= 0 - 2t = -2t \end{aligned}}$$

So,

$$(-2t)\vec{p}_1 + 0 \cdot \vec{p}_2 + t \cdot \vec{p}_3 = \vec{0}$$

for every t.

In particular, for $t = 1$ we get

$$-2\vec{p}_1 + 0\vec{p}_2 + 1 \cdot \vec{p}_3 = \vec{0}.$$

Thus, $\vec{p}_1, \vec{p}_2, \vec{p}_3$ are linearly dependent.

7 (a) Since $\mathbb{R}^3$ has dimension 3, we need 3 vectors to have a basis for $\mathbb{R}^3$. Thus, $\vec{v}_1 = \langle 4, -1, 2 \rangle$, $\vec{v}_2 = \langle -4, 10, 2 \rangle$ are not a basis for $\mathbb{R}^3$.

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7 (b) By problem 6(b) the vectors $\vec{v}_1 = \langle -3, 0, 4 \rangle$, $\vec{v}_2 = \langle 5, -1, 2 \rangle$, $\vec{v}_3 = \langle 1, 1, 3 \rangle$ are linearly independent.

Since we have 3 linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in a vector space $\mathbb{R}^3$ of dimension 3, by a theorem in class they must span $\mathbb{R}^3$ and hence are a basis for $\mathbb{R}^3$.

7 (c) Since $\mathbb{R}^3$ has dimension 3, we need 3 vectors to have a basis for $\mathbb{R}^3$. Thus, $\vec{v}_1 = \langle -2, 0, 1 \rangle$, $\vec{v}_2 = \langle 3, 2, 5 \rangle$, $\vec{v}_3 = \langle 6, -1, 1 \rangle$, $\vec{v}_4 = \langle 7, 0, -2 \rangle$ are not a basis for $\mathbb{R}^3$. We have too many vectors.

[You could also just directly show that these 4 vectors are linearly dependendent and hence not a basis for $\mathbb{R}^3$.]

(8) (a)

The dimension of P_2 is $2+1=3$.

Thus, since we have 3 vectors, the vectors

$\vec{p}_1 = 1$, $\vec{p}_2 = 1+x$, $\vec{p}_3 = 1+x+x^2$

will be a basis if and only if they are linearly independent.

Consider the equation

$$c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3 = \vec{0}$$

which becomes

$$c_1(1) + c_2(1+x) + c_3(1+x+x^2) = \underbrace{0+0x+0x^2}_{\vec{0}}$$

Regrouping we get

$$(c_1+c_2+c_3) + (c_2+c_3)x + c_3 x^2 = 0 + 0x + 0x^2$$

Equating coefficients we get:

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}}$$

This system is already reduced:

$$\boxed{\begin{aligned} c_1 &= -c_2 - c_3 \quad (1) \\ c_2 &= -c_3 \quad (2) \\ c_3 &= 0 \quad (3) \end{aligned}}$$

$\Rightarrow$

(3) gives $c_3 = 0$

(2) gives $c_2 = -c_3 = 0$

(1) gives $c_1 = -c_2 - c_3 = 0 - 0 = 0$

Thus, the only solution to

$$c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3 = \vec{0}$$

is $c_1 = c_2 = c_3 = 0$.

Thus, $\vec{p}_1, \vec{p}_2, \vec{p}_3$ are linearly independent.

Since we have 3 linearly independent vectors in a 3 dimensional space P_2, they are a basis for P_2.

(8) (b) Same idea as 8(a).

The dimension of P_2 is $2+1=3$.

Thus, since we have 3 vectors, the vectors $\vec{p}_1 = 6 - x^2$, $\vec{p}_2 = 1 + x + 4x^2$, $\vec{p}_3 = 8 + 2x + 7x^2$ will be a basis if and only if they are linearly independent.

Consider the equation

$$c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3 = \vec{0}$$

which becomes

$$c_1(6-x^2) + c_2(1+x+4x^2) + c_3(8+2x+7x^2) = 0 + 0x + 0x^2$$

Grouping like terms gives

$$(6c_1 + c_2 + 8c_3) + (c_2 + 2c_3)x + (-c_1 + 4c_2 + 7c_3) = 0 + 0x + 0x^2$$

Equating coefficients gives

$$\boxed{\begin{aligned} 6c_1 + c_2 + 8c_3 &= 0 \\ c_2 + 2c_3 &= 0 \\ -c_1 + 4c_2 + 7c_3 &= 0 \end{aligned}}$$

$\leftarrow$ Let's solve this system

$$\left(\begin{array}{ccc|c} 6 & 1 & 8 & 0 \\ 0 & 1 & 2 & 0 \\ -1 & 4 & 7 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} -1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 6 & 1 & 8 & 0 \end{array}\right)$$

$$\xrightarrow{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & -4 & -7 & 0 \\ 0 & 1 & 2 & 0 \\ 6 & 1 & 8 & 0 \end{array}\right)$$

$$\xrightarrow{-6R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -4 & -7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 25 & 50 & 0 \end{array}\right)$$

$$\xrightarrow{-25R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -4 & -7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The reduced system is:

$$\boxed{\begin{aligned} c_1 - 4c_2 - 7c_3 &= 0 \\ c_2 + 2c_3 &= 0 \\ 0 &= 0 \end{aligned}}$$

Leading variables: c_1, c_2

Free variable: c_3

Solutions:

$$\boxed{\begin{aligned} c_3 &= t \\ c_2 &= -2c_3 = -2t \\ c_1 &= 4c_2 + 7c_3 = 4(-2t) + 7t = -t \end{aligned}}$$

Therefore,

$$(-t)\vec{p}_1 + (-2t)\vec{p}_2 + (t)\vec{p}_3 = \vec{0}$$

for any t.

For example, if $t=1$, then

$$-\vec{p}_1 - 2\vec{p}_2 + \vec{p}_3 = \vec{0}$$

Thus, $\vec{p}_1, \vec{p}_2, \vec{p}_3$ are linearly dependent and hence are <u>not</u> a basis for P_2.

9 (a)

Let's show that the vectors $\langle 1,4\rangle, \langle 3,-2\rangle$ are linearly independent.

Consider the equation

$$c_1\langle 1,4\rangle + c_2\langle 3,-2\rangle = \langle 0,0\rangle.$$

This becomes

$$\langle c_1+3c_2,\ 4c_1-2c_2\rangle = \langle 0,0\rangle$$

which is equivalent to

$$\boxed{\begin{aligned} c_1+3c_2 &= 0 \\ 4c_1-2c_2 &= 0 \end{aligned}}$$

Let's solve this.

$$\left(\begin{array}{cc|c} 1 & 3 & 0 \\ 4 & -2 & 0 \end{array}\right) \xrightarrow{-4R_1+R_2\to R_2} \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -14 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{14}R_2\to R_2} \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & 0 \end{array}\right).$$

So, the reduced system is

$$\boxed{\begin{aligned} c_1+3c_2 &= 0 \\ c_2 &= 0 \end{aligned}} \Longrightarrow$$

which gives

$c_2=0,\ c_1=-3c_2=0.$

Thus, the only solution is $c_1 = c_2 = 0$.

Thus, $\langle 1,4\rangle, \langle 3,-2\rangle$ are linearly independent.

Since $\mathbb{R}^2$ has dimension 2, and we have 2 linearly independent vectors, we can conclude that $\langle 1,4\rangle, \langle 3,-2\rangle$ are a basis for $\mathbb{R}^2$

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9 (b) We must solve

$$\langle -7,14\rangle = c_1\langle 1,4\rangle + c_2\langle 3,-2\rangle$$

which becomes

$$\langle -7,14\rangle = \langle c_1+3c_2,\ 4c_1-2c_2\rangle$$

which becomes

$$\boxed{\begin{aligned} -7 &= c_1+3c_2 \\ 14 &= 4c_1-2c_2 \end{aligned}}$$

Let's solve this system:

$$\left(\begin{array}{cc|c} 1 & 3 & -7 \\ 4 & -2 & 14 \end{array}\right) \xrightarrow{-4R_1+R_2\to R_2} \left(\begin{array}{cc|c} 1 & 3 & -7 \\ 0 & -14 & 42 \end{array}\right) \longrightarrow$$

$$\xrightarrow{-\frac{1}{14}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 3 & -7 \\ 0 & 1 & -3 \end{array}\right)$$

So we get:

$$\boxed{\begin{aligned} c_1 + 3c_2 &= -7 \\ c_2 &= -3 \end{aligned}}$$

So, $c_2 = -3$

$c_1 = -7 - 3c_2 = -7 - 3(-3) = 2$.

Thus,

$$\langle -7, 14 \rangle = 2 \cdot \langle 1, 4 \rangle + (-3) \langle 3, -2 \rangle$$

So, the coordinates of $\langle -7, 14 \rangle$ with respect to the ordered basis $\beta = [\langle 1, 4 \rangle, \langle 3, -2 \rangle]$ are

$$[\langle -7, 14 \rangle]_\beta = \langle 2, -3 \rangle$$

9 (c)

We want to solve

$$\langle 3,-12\rangle = c_1\langle 1,4\rangle + c_2\langle 3,-2\rangle$$

which becomes

$$\langle 3,-12\rangle = \langle c_1+3c_2,\ 4c_1-2c_2\rangle$$

which is equivalent to

$$\boxed{\begin{aligned} 3 &= c_1+3c_2 \\ -12 &= 4c_1-2c_2 \end{aligned}}$$

Let's solve this system:

$$\left(\begin{array}{cc|c} 1 & 3 & 3 \\ 4 & -2 & -12 \end{array}\right) \xrightarrow{-4R_1+R_2\to R_2} \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -14 & -24 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{14}R_2\to R_2} \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 1 & \frac{12}{7} \end{array}\right) \longrightarrow \boxed{\begin{aligned} c_1+3c_2 &= 3 \\ c_2 &= \frac{12}{7} \end{aligned}}$$

$$\longrightarrow c_2 = \frac{12}{7},\quad c_1 = 3-3c_2 = 3-\frac{36}{7} = -\frac{15}{7}$$

Thus,

$$\langle 3,-12\rangle = \left(\frac{-15}{7}\right)\cdot\langle 1,4\rangle + \left(\frac{12}{7}\right)\cdot\langle 3,-2\rangle$$

So, the coordinates of $\langle 3,-12\rangle$ with respect to the ordered basis $\beta = [\langle 1,4\rangle, \langle 3,-2\rangle]$ are

$$[\langle 3,-12\rangle]_\beta = \left\langle \frac{-15}{7}, \frac{12}{7}\right\rangle$$

10 (a)

Let us show that the vectors are linearly independent.

Consider the equation

$$c_1\begin{pmatrix}1&0\\0&1\end{pmatrix}+c_2\begin{pmatrix}1&1\\1&0\end{pmatrix}+c_3\begin{pmatrix}0&0\\0&1\end{pmatrix}+c_4\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

This becomes

$$\begin{pmatrix}c_1+c_2 & c_2-c_4\\ c_2+c_4 & c_1+c_3\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Equating coefficents gives

$$\boxed{\begin{aligned} c_1+c_2 \qquad\qquad &= 0\\ c_2 \qquad -c_4 &= 0\\ c_2 \qquad +c_4 &= 0\\ c_1 \qquad +c_3 \qquad &= 0\end{aligned}}$$

Let's solve this system

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{-R_1 + R_4 \to R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_2 + R_4 \to R_4]{-R_2 + R_3 \to R_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2} R_4 \to R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

This becomes:

$$\boxed{\begin{aligned} c_1 + c_2 \qquad\qquad &= 0 \\ c_2 \qquad - c_4 &= 0 \\ c_3 - c_4 &= 0 \\ c_4 &= 0 \end{aligned}}$$

Solving gives $c_4 = 0$, $c_3 = c_4 = 0$

$c_2 = c_4 = 0$, and $c_1 = -c_2 = -0 = 0$.

Thus, $c_1 = c_2 = c_3 = c_4 = 0$ is the only solution.

So, the vectors

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

are linearly independent.

Since there are 4 of them and the dimension of $M_{2,2}$ is 4, they form a basis for $M_{2,2}$.

(10) (b) We need to solve

$$\begin{pmatrix} 1 & -2 \\ 0 & -3 \end{pmatrix} = c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_4 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which becomes

$$\begin{pmatrix} 1 & -2 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 & c_2 - c_4 \\ c_2 + c_4 & c_1 + c_3 \end{pmatrix}$$

which is equivalent to

$$\boxed{\begin{aligned} c_1 + c_2 &= 1 \\ c_2 - c_4 &= -2 \\ c_2 + c_4 &= 0 \\ c_1 + c_3 &= -3 \end{aligned}}$$

Let's solve this system

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & -3 \end{array}\right) \xrightarrow{-R_1+R_4 \to R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & -4 \end{array}\right)$$

$$\xrightarrow[R_2+R_4 \to R_4]{-R_2+R_3 \to R_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & -1 & -6 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & -6 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_4 \to R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & -6 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

This becomes:

$$\boxed{\begin{aligned} c_1 + c_2 \qquad\qquad &= 1 \\ c_2 \qquad - c_4 &= -2 \\ c_3 - c_4 &= -6 \\ c_4 &= 1 \end{aligned}}$$

Solving this system gives

$c_4 = 1$, $c_3 = -6 + c_4 = -6 + 1 = -5$

$c_2 = -2 + c_4 = -2 + 1 = -1$, and

$c_1 = 1 - c_2 = 1 - (-1) = 2$

Thus,

$$\begin{pmatrix} 1 & -2 \\ 0 & -3 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - 5 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So,

$$\left[\begin{pmatrix} 1 & -2 \\ 0 & -3 \end{pmatrix} \right]_\beta = \langle 2, -1, -5, 1 \rangle$$

(10) (c) We need to solve

$$\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_4 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which becomes

$$\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 & c_2 - c_4 \\ c_2 + c_4 & c_1 + c_3 \end{pmatrix}$$

which is equivalent to

$$\boxed{\begin{array}{rcl} c_1 + c_2 & = & 3 \\ c_2 - c_4 & = & 4 \\ c_2 + c_4 & = & 0 \\ c_1 + c_3 & = & 1 \end{array}}$$

Let's solve this system

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{array}\right) \xrightarrow{-R_1+R_4 \to R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & -2 \end{array}\right)$$

$$\xrightarrow[R_2+R_4 \to R_4]{-R_2+R_3 \to R_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 2 & -4 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & -4 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_4 \to R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

This becomes:

$$\boxed{\begin{aligned} c_1 + c_2 \phantom{{}-c_4} &= 3 \\ c_2 - c_4 &= 4 \\ c_3 - c_4 &= 2 \\ c_4 &= -2 \end{aligned}}$$

Solving this system gives

$c_4 = -2$, $c_3 = 2 + c_4 = 2 - 2 = 0$

$c_2 = 4 + c_4 = 4 - 2 = 2$, and

$c_1 = 3 - c_2 = 3 - 2 = 1$,

Thus,

$$\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; 2 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \; 0 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So,

$$\left[\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}\right]_\beta = \langle 1, 2, 0, -2 \rangle$$

(11) (a) Let $\beta = [1, 1+x, 1+x+x^2]$

We need to solve

$$1 - x + 2x^2 = c_1(1) + c_2(1+x) + c_3(1+x+x^2)$$

which is

$$1 - x + 2x^2 = (c_1 + c_2 + c_3) + (c_2 + c_3)x + c_3 x^2$$

Equating coefficients gives

$$\boxed{\begin{aligned} 1 &= c_1 + c_2 + c_3 \quad (1) \\ -1 &= c_2 + c_3 \quad (2) \\ 2 &= c_3 \quad (3) \end{aligned}}$$

This system is already in reduced form, so we can solve it.

We get $c_3 = 2$, $c_2 = -1 - c_3 = -1 - 2 = -3$,

$c_1 = 1 - c_2 - c_3 = 1 - (-3) - 2 = 2.$

Thus,

$$1 - x + 2x^2 = 2 \cdot (1) - 3 \cdot (1+x) + 2 \cdot (1+x+x^2)$$

So,

$$[1 - x + 2x^2]_\beta = \langle 2, -3, 2 \rangle$$

(11) (b) Let $\beta = [1, 1+x, 1+x+x^2]$

We need to solve

$$x = c_1(1) + c_2(1+x) + c_3(1+x+x^2)$$

which is

$$0 + 1\cdot x + 0\cdot x^2 = (c_1 + c_2 + c_3) + (c_2 + c_3)x + c_3 x^2$$

Equating coefficients gives

$$\boxed{\begin{aligned} 0 &= c_1 + c_2 + c_3 \quad (1) \\ 1 &= c_2 + c_3 \quad (2) \\ 0 &= c_3 \quad (3) \end{aligned}}$$

This system is already in reduced form, so we can solve it.

We get $c_3 = 0$, $c_2 = 1 - c_3 = 1 - 0 = 1$

$c_1 = 0 - c_2 - c_3 = 0 - 1 - 0 = -1$

Thus,

$$x = -1\cdot(1) + 1\cdot(1+x) + 0\cdot(1+x+x^2)$$

So,

$$[x]_\beta = \langle -1, 1, 0 \rangle$$

(12)

I claim that

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for $M_{2,2}$.

If we show this claim then $M_{2,2}$ has dimension 4.

$\boxed{\beta \text{ spans } M_{2,2}}$

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an arbitrary element of $M_{2,2}$.

Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, every element of $M_{2,2}$ is in the span of $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

$\boxed{\beta \text{ is linearly independent}}$

Suppose that

$$c_1\begin{pmatrix}1&0\\0&0\end{pmatrix}+c_2\begin{pmatrix}0&1\\0&0\end{pmatrix}+c_3\begin{pmatrix}0&0\\1&0\end{pmatrix}+c_4\begin{pmatrix}0&0\\0&1\end{pmatrix}=\underbrace{\begin{pmatrix}0&0\\0&0\end{pmatrix}}_{\vec{0}}$$

Then

$$\begin{pmatrix}c_1&c_2\\c_3&c_4\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

So, $c_1=c_2=c_3=c_4=0$.

Thus, $\beta=\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ is a linearly independent set

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Since β is lin. ind. and spans $M_{2,2}$, β is a basis for $M_{2,2}$.

Since β has 4 elements, $M_{2,2}$ has dimension 4.

(13)

$$P_n = \{ a_0 + a_1 \cdot x + a_2 \cdot x^2 + \dots + a_n x^n \mid a_0, a_1, \dots, a_n \in \mathbb{R} \}$$

<u>Claim:</u> $\beta = \{1, x, x^2, \dots, x^n\}$ is a basis for P_n.

$\boxed{\beta \text{ spans } P_n}$:

Let $\vec{p} = a_0 + a_1 x + a_2 x + \dots + a_n x^n$ be an arbitrary element of P_n. Then

$$\vec{p} = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$= a_0 \cdot \underline{1} + a_1 \cdot x + a_2 \cdot x^2 + \dots + a_n x^n$$

So, $\vec{p}$ is in the span of

$\beta = \{1, x, x^2, \dots, x^n\}$

$\boxed{\beta \text{ is a linearly independent set of vectors}}$:

Suppose that

$$c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot x^2 + \cdots + c_n \cdot x^n = \underbrace{0 + 0x + 0x^2 + \cdots + 0x^n}_{\vec{0}}$$

Then equating coefficients gives

$c_0 = 0$, $c_1 = 0$, $c_2 = 0$, $\ldots$, $c_n = 0$.

Thus, $\beta = \{1, x, x^2, \ldots, x^n\}$ is a linearly independent set of vectors.

From the above, $\beta = \{1, x, x^2, \ldots, x^n\}$ is a basis for P_n. Since β has $n+1$ elements in it, the dimension of P_n is $n+1$.