Homework 10 Solutions

① (a) $A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

(i) $\det(A-\lambda I) = \det\begin{pmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{pmatrix} \overset{\text{expand on 1st row}}{=} (4-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix}$

$- 0\begin{vmatrix} -2 & 0 \\ -2 & 1-\lambda \end{vmatrix} + 1\cdot\begin{vmatrix} -2 & 1-\lambda \\ -2 & 0 \end{vmatrix} = (4-\lambda)(1-\lambda)^2 + 1\cdot(-(1-\lambda)(-2))$

$= (1-\lambda)[(4-\lambda)(1-\lambda)+2] = -(\lambda^3 - 6\lambda^2 + 11\lambda - 6) = -(\lambda-1)(\lambda-2)(\lambda-3)$

The eigenvalues are $\lambda = 1, 2, 3$

λ	algebraic multiplicity
1	1
2	1
3	1

(ii)

$\underline{\lambda = 1}$: We need to solve $A\vec{x} = 1\cdot\vec{x}$ or $(A-I)\vec{x} = \vec{0}$.

$$\underbrace{\left(\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right)}_{A-I} \xrightarrow{\frac{1}{3}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right) \xrightarrow[2R_1+R_3 \to R_3]{2R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 2/3 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left.\begin{array}{r} x_1 \quad + \frac{1}{3}x_3 = 0 \\ \frac{2}{3}x_3 = 0 \\ 0 = 0 \end{array}\right\} \quad \begin{array}{l} x_1 = -\frac{1}{3}x_3 = 0 \\ x_2 = t \\ x_3 = 0 \qquad t \in \mathbb{R} \end{array}$$

So, $E_1(A) = \left\{ \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$

A basis for $E_1(A)$ is $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

Thus, $\lambda = 1$ has geometric multiplicity 1.

$\underline{\lambda = 2}$: We need to solve $A\vec{x} = 2\vec{x}$ or $(A - 2I)\vec{x} = \vec{0}$.

$$\left(\underbrace{\begin{matrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{matrix}}_{A-2I} \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right) \xrightarrow[R_1 + R_3 \to R_3]{R_1 + R_2 \to R_2} \left(\begin{matrix} 2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right) \xrightarrow{(-1)R_2 \to R_2} \left(\begin{matrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right)$$

$$\left. \begin{aligned} 2x_1 \quad + x_3 &= 0 \\ x_2 - x_3 &= 0 \\ 0 &= 0 \end{aligned} \right\} \begin{aligned} x_1 &= -\tfrac{1}{2}x_3 \\ x_2 &= x_3 \end{aligned} \Bigg\} \begin{aligned} x_1 &= -\tfrac{1}{2}t \\ x_2 &= t \\ x_3 &= t \end{aligned} \quad t \in \mathbb{R}$$

So, $E_2(A) = \left\{ \begin{pmatrix} -1/2\, t \\ t \\ t \end{pmatrix} \middle| \, t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$

A basis for $E_2(A)$ is $\left\{ \begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix} \right\}$.

So, $\lambda = 2$ has geometric multiplicity 1.

$\underline{\lambda = 3}$: We need to solve $A\vec{x} = 3\vec{x}$ or $(A - 3I)\vec{x} = \vec{0}$.

$$\left(\underbrace{\begin{matrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{matrix}}_{A-3I} \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right) \xrightarrow[2R_1 + R_3 \to R_3]{2R_1 + R_2 \to R_2} \left(\begin{matrix} 1 & 0 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right) \xrightarrow{-\frac{1}{2}R_2 \to R_2} \left(\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right)$$

$$\left. \begin{aligned} x_1 \quad + x_3 &= 0 \\ x_2 - x_3 &= 0 \\ 0 &= 0 \end{aligned} \right\} \begin{aligned} x_1 &= -x_3 = -t \\ x_2 &= x_3 = t \\ x_3 &= t \end{aligned} \quad t \in \mathbb{R}$$

$$E_3(A) = \left\{ \begin{pmatrix} -t \\ t \\ t \end{pmatrix} \middle| \ t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \middle| \ t \in \mathbb{R} \right\}$$

So, a basis for $E_3(A)$ is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Thus, $\lambda = 3$ has geometric multiplicity 1.

(iii)

λ	algebraic multiplicity	geometric multiplicity
1	1	1
2	1	1
3	1	1

(iv) Since the algebraic and geometric multiplicities above are all equal for all λ, A is diagonalizable. The bases of each eigenspace give the columns of P.

Let $P = \begin{pmatrix} 0 & -\frac{1}{2} & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$\boxed{\lambda=1 \quad \lambda=2 \quad \lambda=3 \quad \text{eigenvectors}}$

$\lambda=1 \quad \lambda=2 \quad \lambda=3$

Then, $P^{-1}AP = D$.

① (b) $A = \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix}$

expand on 1st column

(i) $\det(A - \lambda I) = \det\begin{pmatrix} 5-\lambda & 6 & 2 \\ 0 & -1-\lambda & -8 \\ 1 & 0 & -2-\lambda \end{pmatrix} = (5-\lambda)\begin{vmatrix} -1-\lambda & -8 \\ 0 & -2-\lambda \end{vmatrix}$

$- 0 \cdot \begin{vmatrix} 6 & 2 \\ 0 & -2-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 6 & 2 \\ -1-\lambda & -8 \end{vmatrix} = (5-\lambda)(-1-\lambda)(-2-\lambda) + 0 + [(6)(-8) - (2)(-1-\lambda)]$

$= -\lambda^3 + 2\lambda^2 + 15\lambda - 36$

The possible integer/whole number roots of $-\lambda^3 + 2\lambda^2 + 15\lambda - 36$ are the divisors of 36 which are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$. Note that 3 is a root since $-(3^3) + 2(3^2) + 15(3) - 36 = 0$. So we can factor $\lambda - 3$ out of $-\lambda^3 + 2\lambda^2 + 15\lambda - 36$.

$$
\begin{array}{r}
-\lambda^2 - \lambda + 12 \\
\lambda - 3 \enclose{longdiv}{-\lambda^3 + 2\lambda^2 + 15\lambda - 36} \\
-(-\lambda^3 + 3\lambda^2) \\
\hline
-\lambda^2 + 15\lambda - 36 \\
-(-\lambda^2 + 3\lambda) \\
\hline
12\lambda - 36 \\
-(12\lambda - 36) \\
\hline
0
\end{array}
$$

So,

$$
\begin{aligned}
-\lambda^3 + 2\lambda^2 + 15\lambda - 36 &= (\lambda - 3)(-\lambda^2 - \lambda + 12) \\
&= -(\lambda - 3)(\lambda^2 + \lambda - 12) \\
&= -(\lambda - 3)(\lambda - 3)(\lambda + 4) \\
&= -(\lambda - 3)^2(\lambda + 4)
\end{aligned}
$$

So, $\lambda = 3, -4$ are the eigenvalues.

λ	algebraic multiplicity
3	2
-4	1

(iii)

$\underline{\lambda=3}$: Solve $A\vec{x}=3\vec{x}$ or $(A-3I)\vec{x}=\vec{0}$

$$\underbrace{\left(\begin{array}{ccc|c} 2 & 6 & 2 & 0 \\ 0 & -4 & -8 & 0 \\ 1 & 0 & -5 & 0 \end{array}\right)}_{A-3I} \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & -4 & -8 & 0 \\ 2 & 6 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 6 & 12 & 0 \end{array}\right) \xrightarrow[1/6 R_3 \to R_3]{-\frac{1}{4}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{r} x_1 - 5x_3 = 0 \\ x_2 + 2x_3 = 0 \end{array} \Bigg\} \begin{array}{l} x_1 = 5x_3 = 5t \\ x_2 = -2x_3 = -2t \\ x_3 = t \end{array}$$

$$E_3(A) = \left\{ \begin{pmatrix} 5t \\ -2t \\ t \end{pmatrix} \middle| \, t \in \mathbb{R} \right\} = \left\{ t\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$$

So, a basis for $E_3(A)$ is $\left\{ \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \right\}$.

Thus, $\lambda=3$ has geometric multiplicity 1.

$\underline{\lambda=-4}$: Solve $A\vec{x}=-4\vec{x}$ or $(A+4I)\vec{x}=\vec{0}$.

$$\underbrace{\left(\begin{array}{ccc|c} 9 & 6 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)}_{A+4I} \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 9 & 6 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-9R_1+R_3\to R_3}\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 0 & 6 & -16 & 0 \end{array}\right)\xrightarrow{-2R_2+R_3\to R_3}\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{1/3 R_2\to R_2}\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -8/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{r} x_1 \quad +2x_3=0 \\ x_2-\frac{8}{3}x_3=0 \\ 0=0 \end{array}$$

$$x_1 = -2x_3 = -2t$$
$$x_2 = \frac{8}{3}x_3 = \frac{8}{3}t$$
$$x_3 = t$$

$$E_{-4}(A)=\left\{\begin{pmatrix} -2t \\ 8/3t \\ t \end{pmatrix} \middle| t\in\mathbb{R}\right\}=\left\{t\begin{pmatrix} -2 \\ 8/3 \\ 1 \end{pmatrix} \middle| t\in\mathbb{R}\right\}$$

So, a basis for $E_{-4}(A)$ is $\left\{\begin{pmatrix} -2 \\ 8/3 \\ 1 \end{pmatrix}\right\}$.

Thus $\lambda=-4$ has geometric multiplicity 1.

(iii)

λ	algebraic multiplicity	geometric multiplicity
3	2	1
-4	1	1

(iv) A is not diagonalizable. We need a basis of 3 eigenvectors. $\lambda=-4$ gives us enough eigenvectors, but $\lambda=3$ only gives us 1 eigenvector when it should give us 2 if we want to diagonalize A. (Because A has algebraic mult. 2, but geometric mult. 1)

① (c) $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

(i) $\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - (2)(0)$
$= (\lambda-1)(\lambda-3)$

So, the eigenvalues of A are $\lambda = 1, 3$

λ	algebraic multiplicity
1	1
3	1

(ii)

$\underline{\lambda = 1:}$ Solve $A\vec{x} = 1 \cdot \vec{x}$ or $(A - I)\vec{x} = \vec{0}$

$$\underbrace{\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}}_{A-I} \left| \begin{matrix} 0 \\ 0 \end{matrix} \right. \xrightarrow{-R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\left.\begin{matrix} 2x_2 = 0 \\ 0 = 0 \end{matrix}\right\} \begin{matrix} x_1 = t \\ x_2 = 0 \end{matrix}$$

$E_1(A) = \left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} \middle| t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle/ t \in \mathbb{R} \right\}$

$\lambda = 1$ has geometric multiplicity 1

A basis for $E_1(A)$ is $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

$\underline{\lambda = 3}$: Need to solve $A\vec{x} = 3\vec{x}$ or $(A-3I)\vec{x} = \vec{0}$.

$$\underbrace{\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)}_{A-3I} \xrightarrow{-\frac{1}{2}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad \left.\begin{array}{r} x_1 - x_2 = 0 \\ 0 = 0 \end{array}\right\} \begin{array}{l} x_1 = x_2 = t \\ x_2 = t \end{array}$$

$$E_3(A) = \left\{ \begin{pmatrix} t \\ t \end{pmatrix} \middle| \, t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$$

So, a basis for $E_3(A)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$\lambda = 3$ has geometric multiplicity 1.

(iii)

λ	algebraic multiplicity	geometric multiplicity
1	1	1
3	1	1

(IV) We have that the algebraic and geometric multiplicities of each eigenvalue are equal.
So we have a basis of 2 eigenvectors.
That gives us the columns of P to diagonalize A.

$$P = \begin{array}{cc} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ \uparrow \quad \uparrow \\ \lambda=1 \quad \lambda=3 \\ \text{eigenvectors} \end{array} \qquad D = \begin{array}{cc} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \\ \uparrow \quad \uparrow \\ \lambda=1 \quad \lambda=3 \end{array} \qquad P^{-1}AP = D.$$

① (d) $A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

(i) $\det(A-\lambda I)=\det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{pmatrix} \overset{\text{expand on 1st column}}{=} -\lambda\cdot\begin{vmatrix} -\lambda & 2 \\ 0 & -\lambda \end{vmatrix} - 0 + 0$

$= -\lambda\left[(-\lambda)(-\lambda) - (2)(0)\right]$

$= -\lambda^3$

So, $\lambda=0$ is the only eigenvalue of A and has algebraic multiplicity 3.

(ii) $\underline{\lambda=0}$: We need to solve $A\vec{x}=0\vec{x}$ or $A\vec{x}=\vec{0}$.

$$\left(\underbrace{\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{matrix}}_{A} \left| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right.\right) \qquad \left.\begin{matrix} x_2 = 0 \\ 2x_3 = 0 \end{matrix}\right\} \quad \begin{matrix} x_1 = t \\ x_2 = 0 \\ x_3 = 0 \end{matrix}$$

So, $E_0(A)=\left\{\begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \middle| t\in\mathbb{R}\right\} = \left\{t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \middle| t\in\mathbb{R}\right\}$.

So, $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ is a basis for $E_0(A)$. Thus, $\lambda=0$ has geometric multiplicity 1.

(iii)

λ	algebraic multiplicity	geometric multiplicity
0	3	1

(iv) Since $\lambda=0$ has unequal algebraic and geometric multiplicity, A is not diagonalizable.

① (e) $A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$

6

(i) $\det(A-\lambda I) = \det\begin{pmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{pmatrix}$

expand on 1st row

$\overset{\downarrow}{=} (4-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 0 & 4-\lambda \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 1 & 4-\lambda \end{vmatrix} + 1\cdot\begin{vmatrix} 2 & 3-\lambda \\ 1 & 0 \end{vmatrix}$

$= (4-\lambda)(3-\lambda)(4-\lambda) - (3-\lambda) = (3-\lambda)\left[(4-\lambda)(4-\lambda) - 1\right]$

$= (3-\lambda)(\lambda^2 - 8\lambda + 15) = (3-\lambda)(\lambda-5)(\lambda-3) = -(\lambda-3)^2(\lambda-5)$

So, the eigenvalues are $\lambda = 3, 5$.

λ	algebraic multiplicity
3	2
5	1

6

(ii) $\underline{\lambda = 3}$: Need to solve $A\vec{x} = 3\vec{x}$ or $(A-3I)\vec{x} = \vec{0}$.

$\left(\underbrace{\begin{matrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{matrix}}_{A-3I} \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right) \xrightarrow[-R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left(\begin{matrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right)$

$\left.\begin{matrix} x_1 + x_3 = 0 \\ 0 = 0 \\ 0 = 0 \end{matrix}\right\} \begin{matrix} x_1 = -x_3 = -t \\ x_2 = s \\ x_3 = t \end{matrix}$

6

$E_3(A) = \left\{ \begin{pmatrix} -t \\ s \\ t \end{pmatrix} \middle| s, t \in \mathbb{R} \right\} = \left\{ s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}$

So a basis for $E_3(A)$ is $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Thus, $\lambda = 3$ has geometric multiplicity 2.

$\underline{\lambda = 5}$: Need to solve $A\vec{x} = 5\vec{x}$ or $(A - 5I)\vec{x} = \vec{0}$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 2 & -2 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \xrightarrow[R_1 + R_3 \to R_3]{2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow$$

$$\xrightarrow[-\frac{1}{2}R_2 \to R_2]{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left.\begin{array}{r} x_1 \quad -x_3 = 0 \\ x_2 - 2x_3 = 0 \end{array}\right\} \begin{array}{l} x_1 = x_3 = t \\ x_2 = 2x_3 = 2t \\ x_3 = t \end{array}$$

$$E_5(A) = \left\{ \begin{pmatrix} t \\ 2t \\ t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

So a basis for $E_5(A)$ is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$.

Thus $\lambda = 5$ has geometric multiplicity 1.

(iii)

λ	algebraic multiplicity	geometric multiplicity
3	2	2
5	1	1

(iv) Since the algebraic and geometric multiplicities of each eigenvalue are equal we get a basis of eigenvectors to diagonalize A.

Let $P = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

$\lambda=3$ $\lambda=5$ $\lambda=3$ $\lambda=5$

eigenvectors

So, $P^{-1}AP = D$.

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(2) $E_\lambda(A) = \{\vec{x} \mid A\vec{x} = \lambda\vec{x} \text{ and } \vec{x} \in \mathbb{R}^n\}$

- Note that $A\vec{0} = \vec{0} = \lambda \cdot \vec{0}$. Thus, $\vec{0}$ is in $E_\lambda(A)$. since $A\vec{0} = \lambda\vec{0}$.
- Suppose that $\vec{x}_1$ and $\vec{x}_2$ are in $E_\lambda(A)$.

Then by the definition of $E_\lambda(A)$ we know that $A\vec{x}_1 = \lambda\vec{x}_1$ and $A\vec{x}_2 = \lambda\vec{x}_2$.

Thus, $A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \lambda\vec{x}_1 + \lambda\vec{x}_2 = \lambda(\vec{x}_1 + \vec{x}_2)$.

Since $A(\vec{x}_1 + \vec{x}_2) = \lambda(\vec{x}_1 + \vec{x}_2)$ we know that $\vec{x}_1 + \vec{x}_2$ is in $E_\lambda(A)$.

- Let $\vec{x}$ be in $E_\lambda(A)$ and c be in $\mathbb{R}$.

Then $A\vec{x} = \lambda\vec{x}$ since x is in $E_\lambda(A)$.

Multiplying by c gives $c(A\vec{x})+c(\lambda\vec{x})$,

Thus, $A(c\vec{x})=\lambda(c\vec{x})$.

So, $c\vec{x}$ is in $E_\lambda(A)$.

Thus, $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.

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③ Since λ is an eigenvalue of A with corresponding eigenvector $\vec{x}$ we have that $\boxed{A\vec{x}=\lambda\vec{x},}$

Multiplying both sides by A gives

$$A^2\vec{x}=A(\lambda\vec{x})$$
$$=\lambda(A\vec{x})=\lambda(\lambda\vec{x})=\lambda^2\vec{x}.$$

So, $\boxed{A^2\vec{x}=\lambda^2\vec{x}}$

Multiplying this equation by A gives

$$A^3\vec{x}=A(\lambda^2\vec{x})$$
$$=\lambda^2(A\vec{x})=\lambda^2(\lambda\vec{x})=\lambda^3\vec{x}$$

So, $\boxed{A^3\vec{x}=\lambda^3\vec{x}}$

Carrying on in this fashion gives

$A^n\vec{x}=\lambda^n\vec{x}$ for all $n=1,2,3,4,\ldots$