Math 2550
9/27/23

# Topic 4 - Inverse of a matrix

In the real numbers, every non-zero number has a multiplicative inverse. For example, $3 \cdot \frac{1}{3} = 1$.

We write $3^{-1} = \frac{1}{3}$.

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Can we do this for matrices?

We will look at square matrices. $\Big\}$ $n \times n$

<u>Def:</u> Let $A$ be an $n \times n$ matrix. [So, $A$ is a square matrix.]

We say that $A$ is <u>invertible</u> if there exists an $n \times n$ matrix $B$ where

$$\boxed{AB = BA = I_n}$$

$n \times n$ identity matrix

If $AB = BA = I_n$, then we say that $A$ and $B$ are <u>inverses</u> of each other.

Ex: Let

$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$

Let's check if $A$ and $B$ are inverses of each other.

$$AB = \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}}_{2\times 2}\underbrace{\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}}_{2\times 2}$$

✓ outcome is $2\times 2$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} & \begin{pmatrix} 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -1+2 & 1-1 \\ -2+2 & 2-1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

And

$$BA = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1+2 & -1+1 \\ 2-2 & 2-1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

We just showed that A and B are inverses of each other.

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<u>Theorem:</u> Suppose that A is an nxn matrix that is invertible, ie an inverse for A exists. Then there exists only <u>one</u> nxn matrix B where $AB = BA = I_n$.

[That is, if A has an inverse, there is only one inverse for A].

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Notation: If $A$ is an $n \times n$ matrix that is invertible then we write $A^{-1}$ for the inverse of $A$.

Ex: $$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

from the previous calculation

## $\boxed{\text{How to find } A^{-1} \text{ for a square matrix } A\text{, if it exists}}$

Let $A$ be an $n \times n$ matrix

① $A^{-1}$ exists if and only if one can row reduce $A$ down to $I_n$

② Procedure: Start with the matrix $(A \mid I_n)$. Do row reduction on this matrix until the left side is either $I_n$ or has a row of zeros.

If you end up with a row of zeros on the left side, then $A^{-1}$ does not exist.

If you end up with $I_n$ on the left side, then $A^{-1}$ exists and its the matrix on the right side.

$\underline{\text{Ex:}}$ Find $A^{-1}$ if it exists

if $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$.

$\boxed{2 \times 2}$

$\boxed{\text{use } I_2}$

already 1

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right)$$

$$\underbrace{\phantom{1 \quad 1}}_{A} \qquad \underbrace{\phantom{1 \quad 0}}_{I_2}$$

$\boxed{\text{make } 0}$

$\boxed{\text{now row reduce the left side to reduced row echelon form}}$

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right)$$

now make this 1

make this 0

$$\xrightarrow{-R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

$$\xrightarrow{-R_2+R_1 \to R_1} \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{\text{this is } I_2} \qquad \underbrace{\begin{matrix} -1 & 1 \\ 2 & -1 \end{matrix}}_{\text{this is } A^{-1}}$$

So, $A^{-1}$ exists and $A^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$

Ex: Find $A^{-1}$ if it exists

where $A = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}$. $\leftarrow \boxed{3\times 3}$

use $I_3$

make this 1

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

$\underbrace{\qquad}_{A}$ $\underbrace{\qquad}_{I_3}$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 3 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

use the 1 to make these 0

$$\xrightarrow[2R_1+R_3\to R_3]{-3R_1+R_2\to R_2}\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -3 & 1 & -3 & 0 \\ 0 & 5 & 4 & 0 & 2 & 1 \end{array}\right)$$

make this 1

$$\xrightarrow{-\frac{1}{3}R_2\to R_2}\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1/3 & 1 & 0 \\ 0 & 5 & 4 & 0 & 2 & 1 \end{array}\right)$$

make into 0 using the 1 we made

$$\xrightarrow[-5R_2+R_3\to R_3]{-R_2+R_1\to R_1}\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/3 & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & -1 & 5/3 & -3 & 1 \end{array}\right)$$

make this 1

$$\xrightarrow{-R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/3 & 0 & 0 \\ 0 & 1 & 1 & -1/3 & 1 & 0 \\ 0 & 0 & 1 & -5/3 & 3 & -1 \end{array}\right)$$

make these 0

$$\xrightarrow[-R_3+R_2 \to R_2]{-R_3+R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 1 \\ 0 & 1 & 0 & 4/3 & -2 & 1 \\ 0 & 0 & 1 & -\frac{5}{3} & 3 & -1 \end{array}\right)$$

$\underbrace{\qquad}_{I_3}$ $\underbrace{\qquad}_{A^{-1}}$

So, $A^{-1}$ exists and

$$A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$$