Math 2550

9/25/23

Ex: (continued from last time...)

We were solving

$$\boxed{\begin{aligned} -2b+3c&=1\\ 3a+6b-3c&=-2\\ 6a+6b+3c&=5 \end{aligned}}$$

We did this:

$$\left(\begin{array}{ccc|c} 0 & -2 & 3 & 1\\ 3 & 6 & -3 & -2\\ 6 & 6 & 3 & 5 \end{array}\right) \xrightarrow{\text{(stuff)}} \cdots \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3\\ 0 & 1 & -3/2 & -1/2\\ 0 & 0 & 0 & 6 \end{array}\right)$$

in row echelon form

Turn back to equations:

$$\boxed{\begin{aligned} a+2b-c&=-2/3\\ b-\tfrac{3}{2}c&=-1/2\\ 0&=6 \end{aligned}}$$

no solutions to the system

Ex: Solve

$$\boxed{\begin{aligned} 5x_1 - 2x_2 + 6x_3 &= 0 \\ -2x_1 + x_2 + 3x_3 &= 1 \end{aligned}}$$

make this into a 1

$$\left(\begin{array}{ccc|c} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{array}\right) \xrightarrow{2R_2 + R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ -2 & 1 & 3 & 1 \end{array}\right)$$

make this into 0

could have instead done $\frac{1}{5}R_1 \to R_1$

$$\xrightarrow{2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ \underbrace{0 \quad 1 \quad 27}_{\text{row echelon form}} & & & 5 \end{array}\right)$$

This becomes:

$$\boxed{\begin{aligned} x_1 \qquad\quad + 12x_3 &= 2 \quad (1) \\ x_2 + 27x_3 &= 5 \quad (2) \end{aligned}}$$

leading variables: x_1, x_2

free variable: x_3

Solve equations for leading variables and give free variables new names:

$$\boxed{\begin{aligned} x_1 &= 2 - 12x_3 \quad (1) \\ x_2 &= 5 - 27x_3 \quad (2) \\ x_3 &= t \quad (3) \end{aligned}}$$

Now back substitute:

(3) $x_3 = t$

(2) $x_2 = 5 - 27x_3 = 5 - 27t$

(1) $x_1 = 2 - 12x_3 = 2 - 12t$

<u>Answer:</u> The solutions to the system are:

$$x_1 = 2 - 12t$$
$$x_2 = 5 - 27t$$
$$x_3 = t$$

where t can be any real number.

What does this mean?
For example if $t = 0$
then you get

$x_1 = 2$, $x_2 = 5$, $x_3 = 0$

So, $(x_1, x_2, x_3) = (2, 5, 0)$ is a solution to the system.

If $t = 1$, then

$(x_1, x_2, x_3) = (-10, -22, 1)$

is another solution to the system.

There are an infinite # of solutions, one for each t.

$\boxed{\text{Ex:}}$ Solve

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \phantom{{}- 2x_4} + 2x_5 \phantom{{}- 3x_6} &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 \phantom{{}+ 4x_5} + 15x_6 &= 5 \\
2x_1 + 6x_2 \phantom{{}- 5x_3} + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}$$

already a 1

$$\left(\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
2 & 6 & -5 & -2 & 4 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{array}\right)$$

make these 0

$$\xrightarrow{-2R_1+R_2\to R_2}\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_4\to R_4}\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

make this a 1

$$\xrightarrow{-R_2\to R_2}\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

make these into 0's

$$\xrightarrow[-4R_2+R_4 \to R_4]{-5R_2+R_3 \to R_3} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

make into 1

$$\xrightarrow{\frac{1}{6}R_3 \to R_3} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

in row echelon form

Put back into equations:

$$x_1 + 3x_2 - 2x_3 \qquad + x_5 \qquad = 0 \quad (1)$$
$$x_3 + 2x_4 \qquad + 3x_6 = 1 \quad (2)$$
$$x_6 = 1/3 \quad (3)$$
$$0 = 0 \quad (4)$$

leading variables: x_1, x_3, x_6

free variables: x_2, x_4, x_5

Solve equations for leading variables and give free variables new names.

$$x_1 = -3x_2 + 2x_3 - x_5 \quad (1)$$
$$x_3 = 1 - 2x_4 - 3x_6 \quad (2)$$
$$x_6 = 1/3 \quad (3)$$
$$x_2 = a \quad (4)$$
$$x_4 = b \quad (5)$$

$$\boxed{x_5 = c} \quad ⑥$$

Back-substitute:

⑥ $x_5 = c$

⑤ $x_4 = b$

④ $x_2 = a$

③ $x_6 = 1/3$

② $x_3 = 1 - 2x_4 - 3x_6 = 1 - 2b - 3\left(\frac{1}{3}\right)$

$= -2b$

① $x_1 = -3x_2 + 2x_3 - x_5$

$= -3a + 2(-2b) - c$

$= -3a - 4b - c$

Answer:

$x_1 = -3a - 4b - c$

$x_2 = a$

$x_3 = -2b$

$x_4 = b$

$x_5 = c$

$x_6 = 1/3$

where a, b, c can be any real #s

Ex: $a=1, b=1, c=0$ gives the sol:

$(x_1, x_2, x_3, x_4, x_5, x_6) = (-7, 1, -2, 1, 0, 1/3)$

There are infinitely many more

Theorem: A system of linear equations has either

(i) no solutions

(ii) exactly one solution

or (iii) an infinite # of solutions