Math 2550
9/18/23

Ex: (Interchanging two rows/equations)

Equation viewpoint:

$$\boxed{\begin{aligned} 3x - y + z &= 1 \\ 5x \quad + 2z &= 2 \\ x + y + z &= -1 \end{aligned}} \xrightarrow{R_1 \leftrightarrow R_3} \boxed{\begin{aligned} x + y + z &= -1 \\ 5x \quad + 2z &= 2 \\ 3x - y + z &= 1 \end{aligned}}$$

augmented matrix viewpoint:

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 5 & 0 & 2 & 2 \\ 1 & 1 & 1 & -1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 5 & 0 & 2 & 2 \\ 3 & -1 & 1 & 1 \end{array}\right)$$

Ex: (Adding a multiple of one row/equation to another row/equation)

Equation viewpoint

$$\boxed{\begin{aligned} x - y + z &= 3 \\ 2x + y + z &= 1 \\ y + 2z &= 10 \end{aligned}} \xrightarrow{-2R_1 + R_2 \to R_2} \boxed{\begin{aligned} x - y + z &= 3 \\ 3y - z &= -5 \\ y + 2z &= 10 \end{aligned}}$$

$$\boxed{\begin{array}{rrrrrrrl} & -2x & + 2y & - 2z & = & -6 & \leftarrow & \boxed{-2R_1} \\ + & 2x & + y & + z & = & 1 & \leftarrow & \boxed{R_2} \\ \hline & & 3y & - z & = & -5 & \leftarrow & \boxed{\text{new } R_2} \end{array}}$$

Augmented matrix viewpoint:

$$\left(\begin{array}{rrr|r} 1 & -1 & 1 & 3 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 10 \end{array}\right) \xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{rrr|r} 1 & -1 & 1 & 3 \\ 0 & 3 & -1 & -5 \\ 0 & 1 & 2 & 10 \end{array}\right)$$

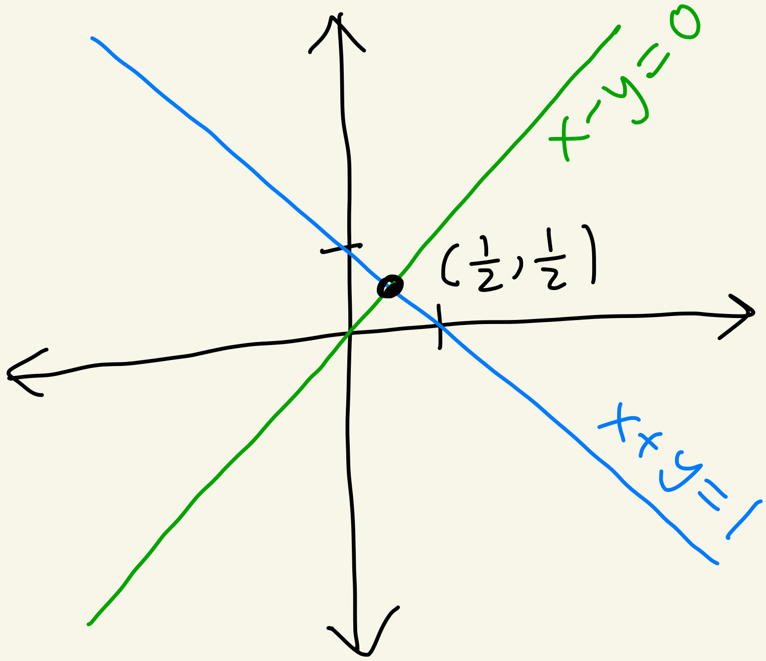
$$\boxed{\begin{array}{rl} & \left(\begin{array}{rrr|r} -2 & 2 & -2 & -6 \end{array}\right) \leftarrow \boxed{-2R_1} \\ + & \left(\begin{array}{rrr|r} 2 & 1 & 1 & 1 \end{array}\right) \leftarrow \boxed{R_2} \\ \hline & \left(\begin{array}{rrr|r} 0 & 3 & -1 & -5 \end{array}\right) \leftarrow \boxed{\text{new } R_2} \end{array}}$$

Theorem: Applying an elementary row operation to a system of linear equations does not change the solution space of the system.

Ex:

system:

$$\boxed{\begin{aligned} x+y&=1 \\ x-y&=0 \end{aligned}} \quad (*)$$



Solution space is just one point
$(x,y) = (\frac{1}{2}, \frac{1}{2})$

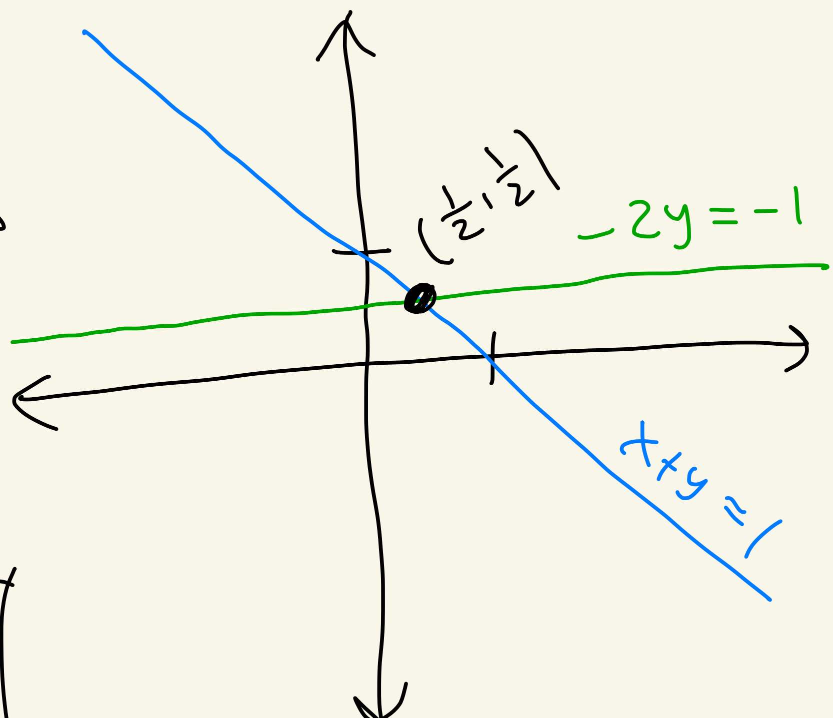
Let's do an elementary row operation to (*)

$$\begin{cases} x+y=1 \\ x-y=0 \end{cases} \xrightarrow{-R_1+R_2 \to R_2} \begin{cases} x+y=1 \\ -2y=-1 \end{cases} \quad (**)$$

$$\begin{array}{rll} & -x-y=-1 & \leftarrow \boxed{-R_1} \\ + & x-y=0 & \leftarrow \boxed{R_2} \\ \hline & -2y=-1 & \leftarrow \boxed{\text{new } R_2} \end{array}$$

Solution space to (**) is

$(x,y)=\left(\frac{1}{2},\frac{1}{2}\right)$



The same solution space

Def: If a row of a matrix does not consist entirely of zeros then the **leading entry** in that row is the first non-zero entry when scanning from left to right.

Ex:

$$A = \begin{array}{c} \begin{pmatrix} 5 & 0 & 2 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & -3 & 1 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \leftarrow \text{row } 3 \\ \leftarrow \text{row } 4 \end{array}$$

leading entry of row 1 is 5
leading entry of row 2 is −1
leading entry of row 3 is −3
No leading entry of row 4

Def: A matrix is in <u>row echelon form</u> if the following are true:

1. If there are any rows that consist entirely of zeros, then those rows are grouped together at the bottom of the matrix.
2. In any two consecutive rows that do not consist entirely of zeros, then the leading entry in the lower row occurs farther to the right than the leading entry in the higher row.

and

3. if a row does not consist entirely of zeros, then the leading entry of that row is 1.

A matrix is in <u>reduced row echelon form</u> if 1, 2, and 3 above are true and:

4. Each column that contains a leading 1 has zeros everywhere else in that column.

Ex:

$$\begin{pmatrix} 1 & 5 & 10 & 7 \\ 0 & -2 & 1 & \frac{1}{2} \end{pmatrix}$$

not a 1

① ✓
② ✓
③ ✗

matrix is not in row echelon form

Ex:

$$\begin{pmatrix} 1 & 5 & 10 & 7 \\ 0 & 1 & 1 & 1/2 \end{pmatrix}$$

columns with leading 1's

① ✓
② ✓
③ ✓

matrix is in row echelon form

④ ✗ not in reduced row echelon form

because of the 5

Ex:

$$\begin{pmatrix} 1 & 0 & 10 & 7 \\ 0 & 1 & 1 & 1/2 \end{pmatrix}$$

① ✓
② ✓
③ ✓

in row echelon form

④ ✓ is in reduced row echelon form

Ex:

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

① ✓
② ✗
③ ✓

not in row echelon form

also not in reduced row echelon form

④ ✓

Ex:

$$\begin{array}{c} \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccccc} & \checkmark & & \times & \times \end{array} \end{array}$$

① ✓
② ✓
③ ✓

in row echelon form

④ ✗

not in reduced row echelon form