Math 2550
9/13/23

Topic 3 - Systems of linear equations

Def: A linear equation in the n variables $x_1, x_2, \dots, x_n$ is an equation of the form

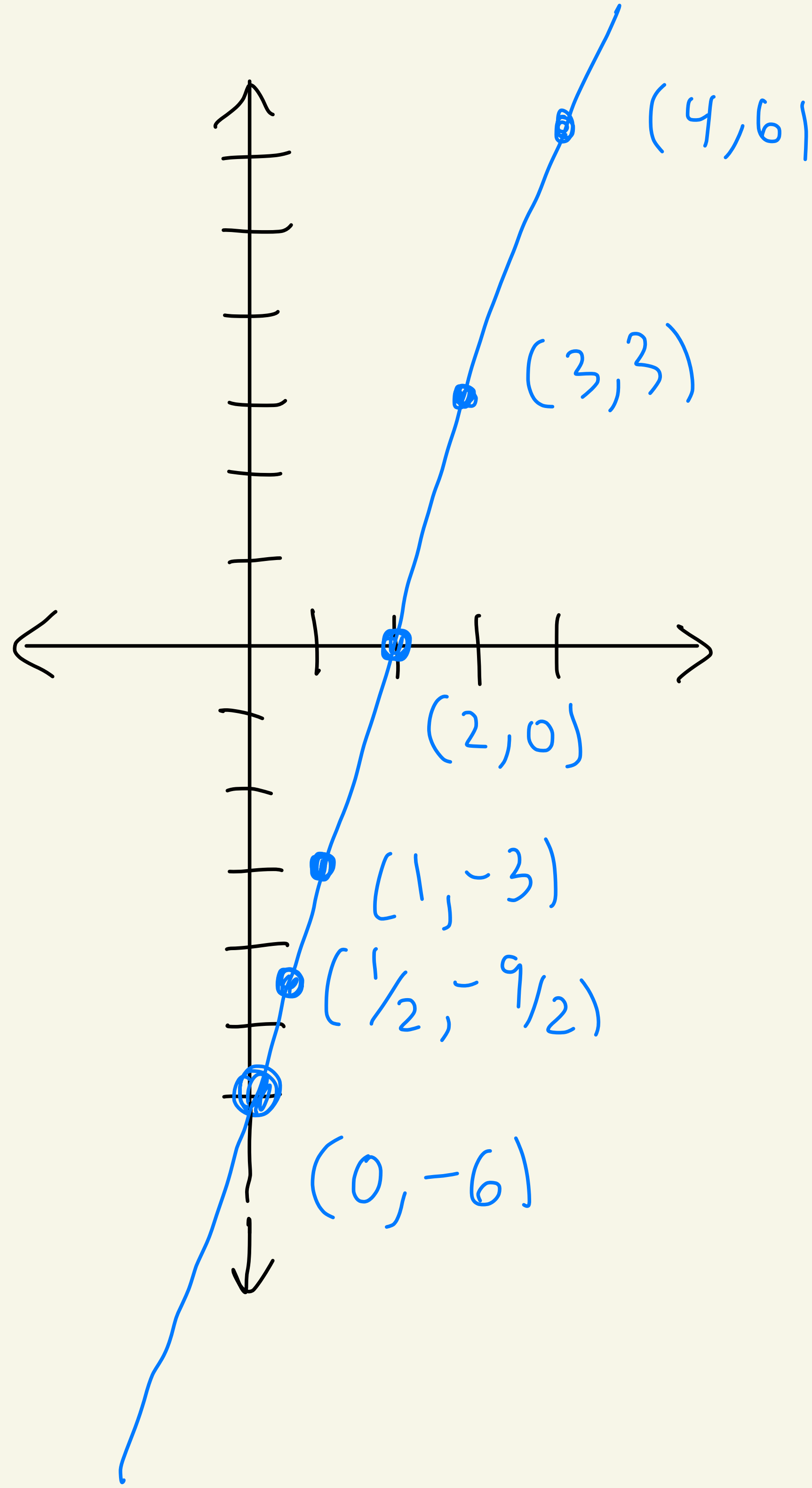
$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b \qquad (*)$$

where $a_1, a_2, \dots, a_n, b$ are constant real numbers.

The solution space of the above equation $(*)$ consists of the set of all $(x_1, x_2, \dots, x_n)$ that solve the equation.

Ex: $3x - y = 6$ ← linear

$\boxed{y = 3x - 6}$



picture of solution space of $3x - y = 6$

$$\frac{3}{2} - 6 = \frac{3-12}{2} = \frac{-9}{2}$$

Another way to describe the solution space to $3x-y=6$ is as the following set:

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, 3x-y=6 \text{ and } x,y\in\mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ 3x-6 \end{pmatrix} \,\middle|\, x\in\mathbb{R} \right\}$$

$$= \left\{ \underbrace{\begin{pmatrix} 0 \\ -6 \end{pmatrix}}_{x=0}, \underbrace{\begin{pmatrix} 10 \\ 24 \end{pmatrix}}_{x=10}, \underbrace{\begin{pmatrix} \pi \\ 3\pi-6 \end{pmatrix}}_{x=\pi}, \dots \right\}$$

infinitely many more

Examples of linear equations:

$$10z - \frac{1}{2}y + 13x = 0$$

$$\frac{1}{2}x_1 + \frac{1}{4}x_2 + \frac{1}{8}x_3 + \frac{1}{16}x_4 = 2$$

Examples of non-linear equations:

$$x^2 - y = 10$$

$$x + \sin(y) - z = x^3$$

Def: A system of m linear equations in the n unknowns $x_1, x_2, \ldots, x_n$ is a set of m equations of the form

$$\boxed{\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}} \quad (*)$$

Where the a_{ij} are constant real number.

The augmented matrix for $(*)$ is

$$\begin{array}{c}
\left(\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right) \\
\begin{array}{cccc}
x_1 \text{ column} & x_2 \text{ column} & x_n \text{ column} & \text{represents } = \text{ sign}
\end{array}
\end{array}$$

The <u>solution space</u> of the system (*) consists of the set of all $(x_1, x_2, \ldots, x_n)$ that simultaneously solve all m equations. That is, the common solutions to all m equations.

Ex:

$$\boxed{\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned}}$$

$m = 2$ equations

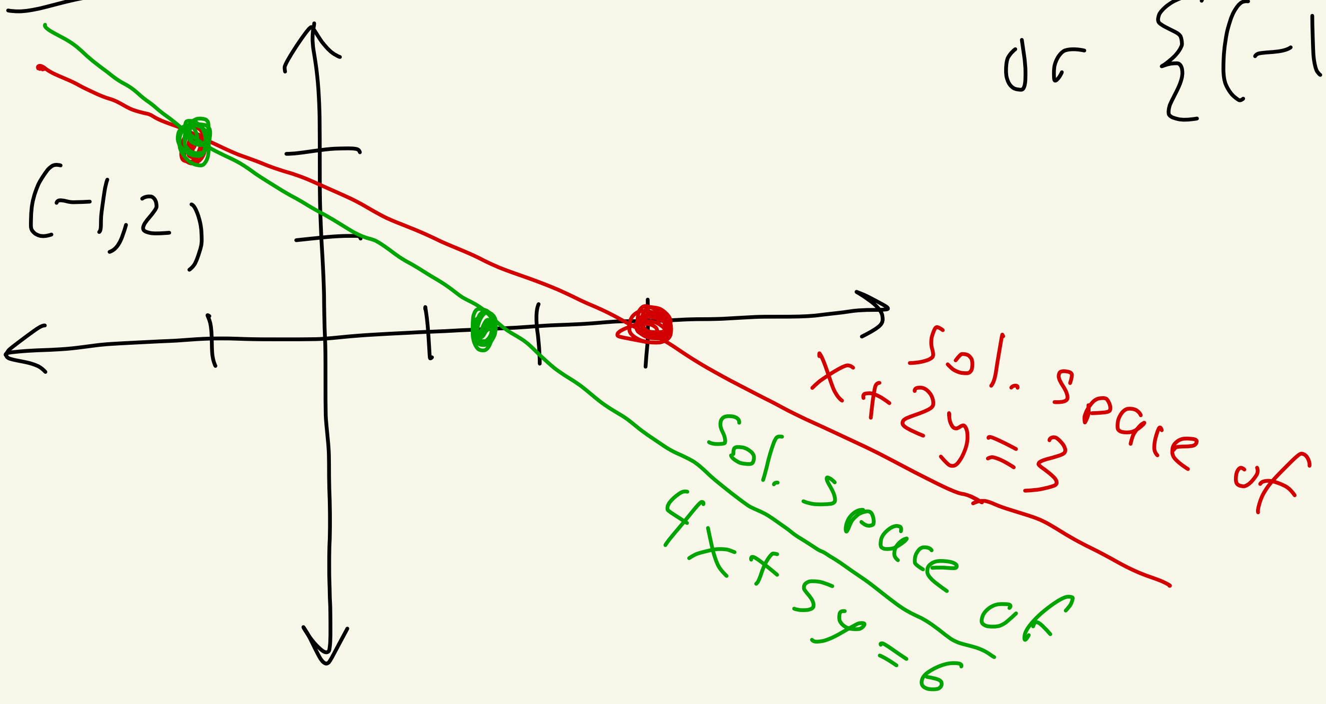
$n = 2$ unknowns

augmented matrix

$$\begin{array}{cc} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right) \\ \begin{array}{cc} x \text{ column} & y \text{ column} \end{array} \end{array}$$

solution space

The solution space of the system is just one point $(x,y) = (-1,2)$ or $\{(-1,2)\}$



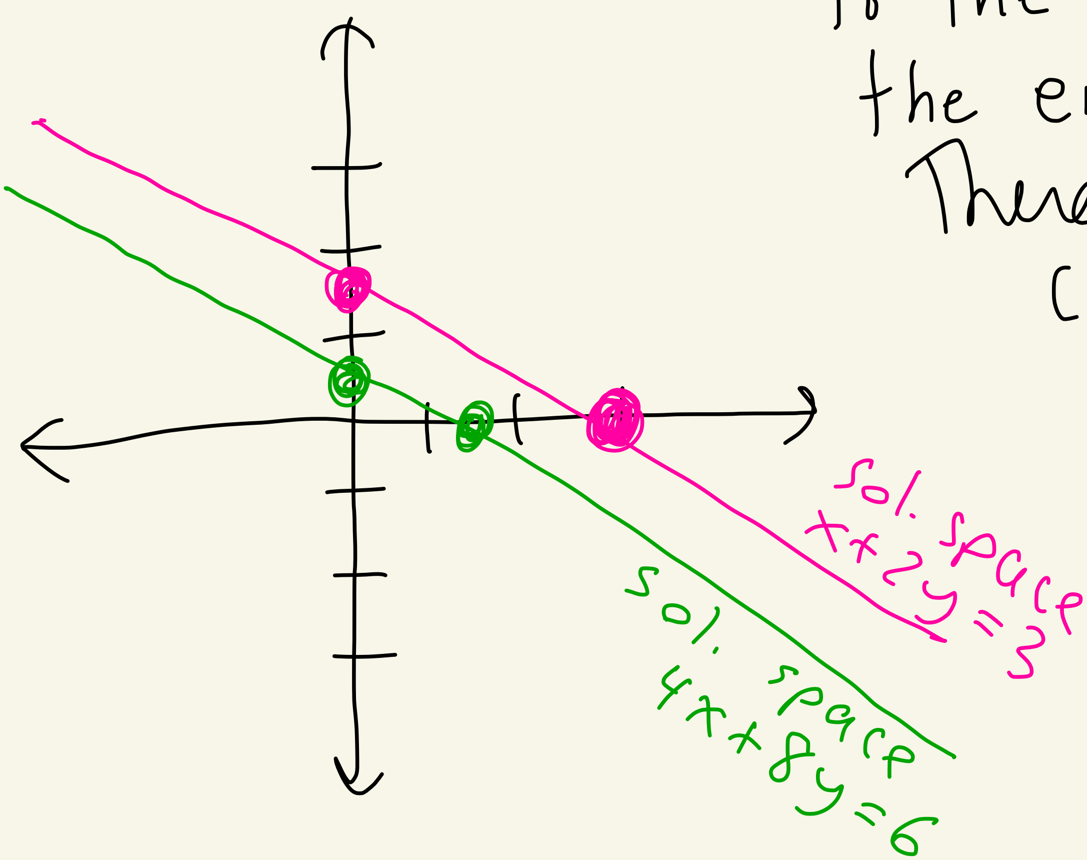
Ex:

$$\boxed{\begin{aligned} x+2y &= 3 \\ 4x+8y &= 6 \end{aligned}}$$

$m=2$ equations
$n=2$ unknowns

$\boxed{\text{augmented matrix:}}$ $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 6 \end{array}\right)$

$\boxed{\text{solution space:}}$



The solution space to the system is the empty set. There are no common solutions to the two equations

Ex:

$$\boxed{\begin{aligned} x + 2y &= 3 \\ 4x + 8y &= 12 \end{aligned}}$$

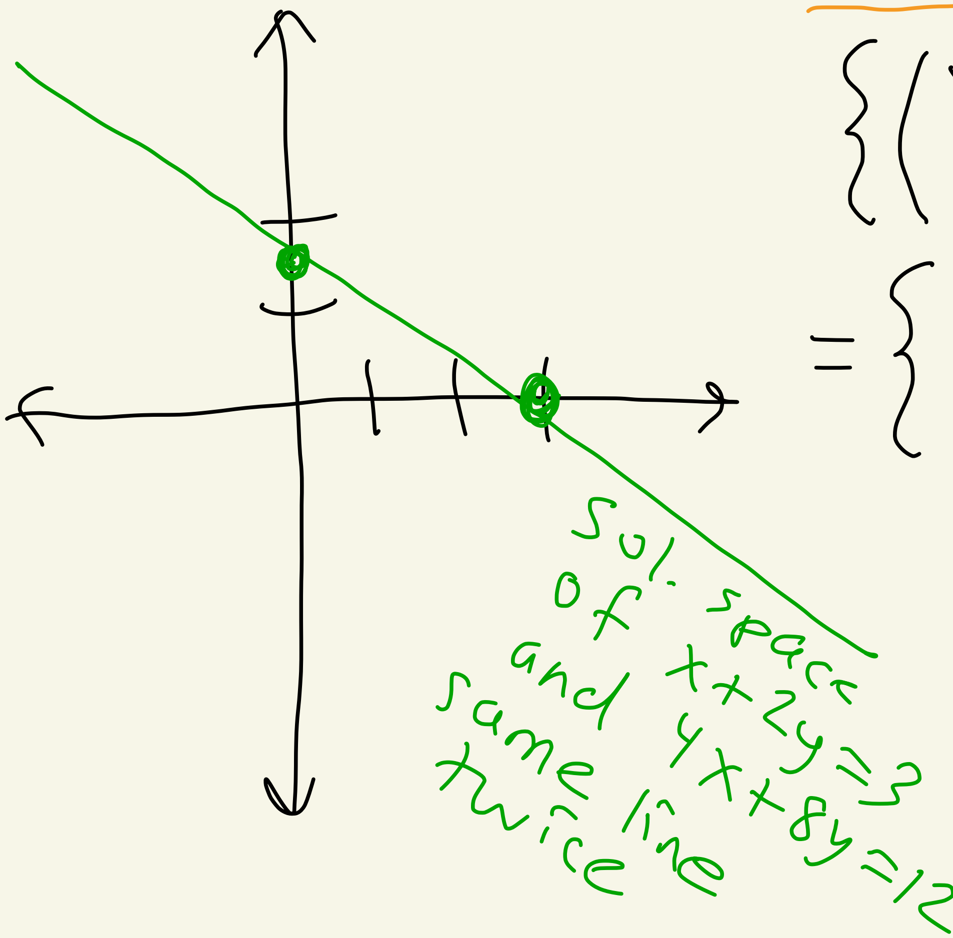
$m = 2$ equations
$n = 2$ unknowns

augmented matrix:
$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 12 \end{array}\right)$$

Solution space:

Solution space is infinite. It is

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \, x + 2y = 3 \right\}$$
$$= \left\{ \begin{pmatrix} x \\ -\frac{1}{2}x + \frac{3}{2} \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$$



Ex:

$$\boxed{\begin{aligned} x + y + 2z &= 9 \\ 2x \qquad - 3z &= 1 \\ -x + 6y - 5z &= 0 \end{aligned}}$$

$m = 3$ equations

$n = 3$ unknowns

Augmented matrix

$$\begin{array}{ccc}
\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 0 & -3 & 1 \\ -1 & 6 & -5 & 0 \end{array}\right) \\
\begin{array}{ccc} x \text{ column} & y \text{ column} & z \text{ column} \end{array}
\end{array}$$

Solution space: Intersection of 3 planes in 3d

We won't solve today but later we will find out that the sol. space is just one point $(x, y, z) = (1, 2, 3)$.

We are going to learn something called **Gaussian elimination** or **row reduction**.

We need a bunch of tools and stuff first.

Def: Given a system of linear equations there are three **elementary row operations**:

1. Multiply one of the rows/equations by a non-zero constant.
2. Interchange two rows/equations.

(3) Add a multiple of one row/equation to a different row/equation.

Ex: (multiply a row/equation by a non-zero constant)

Equation viewpoint

$$\begin{aligned} 3x - y + z &= 1 \\ 5x \quad + 2z &= 2 \\ x + y + z &= -1 \end{aligned} \xrightarrow{\frac{1}{3}R_1 \to R_1} \begin{aligned} x - \tfrac{1}{3}y + \tfrac{1}{3}z &= \tfrac{1}{3} \\ 5x \quad + 2z &= 2 \\ x + y + z &= -1 \end{aligned}$$

Augmented matrix viewpoint:

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 5 & 0 & 2 & 2 \\ 1 & 1 & 1 & -1 \end{array}\right) \xrightarrow{\frac{1}{3}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 5 & 0 & 2 & 2 \\ 1 & 1 & 1 & -1 \end{array}\right)$$