

Math 2550
8/30/23



proof of property ③

when $n=2$

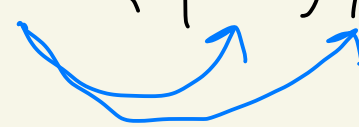
$$\alpha(\beta \vec{u}) = (\alpha\beta) \vec{u}$$

proof: Let $\vec{u}$ be a vector in $\mathbb{R}^2$. Let α, β be in $\mathbb{R}$.

Then, $\vec{u} = \langle x, y \rangle$ where x and y are in $\mathbb{R}$.

So,

$$\alpha(\beta \vec{u}) = \alpha(\beta \langle x, y \rangle)$$


$$= \alpha \langle \beta x, \beta y \rangle$$


$$= \langle \alpha\beta x, \alpha\beta y \rangle$$

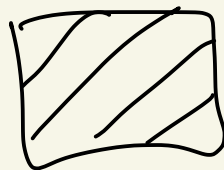

Also,

$$\begin{aligned}(\alpha\beta)\vec{u} &= (\alpha\beta)\langle x, y \rangle \\ &= \langle \alpha\beta x, \alpha\beta y \rangle\end{aligned}$$

the same

Therefore,

$$\alpha(\beta\vec{u}) = (\alpha\beta)\vec{u}.$$



Let's now prove (5) when $n=3$. $\left[\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v} \right]$

proof: Let α be a real number. Let $\vec{u}, \vec{v}$ be in $\mathbb{R}^3$.

Then, $\vec{u} = \langle x, y, z \rangle$

$$\text{and } \vec{v} = \langle a, b, c \rangle$$

where x, y, z, a, b, c are real #s.

Then,

$$\alpha(\vec{u} + \vec{v})$$

$$= \alpha(\langle x, y, z \rangle + \langle a, b, c \rangle)$$

$$= \alpha\langle x+a, y+b, z+c \rangle$$

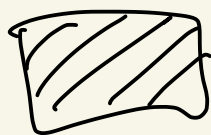
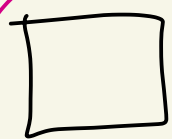
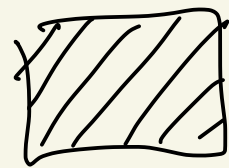

$$= \langle \alpha(x+a), \alpha(y+b), \alpha(z+c) \rangle$$

$$= \langle \alpha x + \alpha a, \alpha y + \alpha b, \alpha z + \alpha c \rangle$$

$$= \langle \alpha x, \alpha y, \alpha z \rangle + \langle \alpha a, \alpha b, \alpha c \rangle$$

$$= \alpha\langle x, y, z \rangle + \alpha\langle a, b, c \rangle$$

$$= \alpha \vec{u} + \alpha \vec{v}.$$



QED

(end of proof markers)

Def: Let $\vec{v}$ and $\vec{w}$ be in $\mathbb{R}^n$.

where $\vec{v} = \langle a_1, a_2, \dots, a_n \rangle$

and $\vec{w} = \langle b_1, b_2, \dots, b_n \rangle$

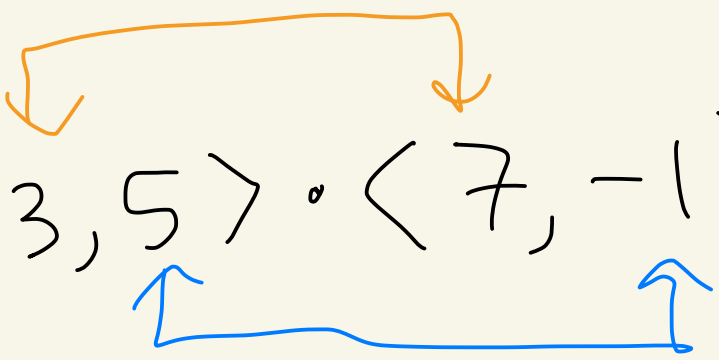
The dot product of $\vec{v}$ and $\vec{w}$
is defined to be

$$\vec{v} \cdot \vec{w} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

Note that $\vec{v} \cdot \vec{w}$ is a number.

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle 3, 5 \rangle$
and $\vec{w} = \langle 7, -1 \rangle$.

Then,

$$\vec{v} \cdot \vec{w} = \langle 3, 5 \rangle \cdot \langle 7, -1 \rangle$$


$$= (3)(7) + (5)(-1)$$

$$= 21 - 5$$

$$= 16$$

Ex: In $\mathbb{R}^5$, let

$$\vec{v} = \langle 1, 1, 7, -5, 2 \rangle \text{ and}$$
$$\vec{w} = \langle 0, \frac{1}{2}, -1, 5, 8 \rangle.$$

Then,

$$\vec{v} \cdot \vec{w} = \langle 1, 1, 7, -5, 2 \rangle \cdot$$
$$\langle 0, \frac{1}{2}, -1, 5, 8 \rangle$$

$$= (1)(0) + (1)(\frac{1}{2}) + (7)(-1)$$
$$+ (-5)(5) + (2)(8)$$

$$= 0.5 - 7 - 25 + 16$$

$$= -16 + 0.5 = -15.5$$

Properties of the dot product

Let $\vec{u}, \vec{v}, \vec{w}$ be in $\mathbb{R}^n$.

Let α be in $\mathbb{R}$.

Then:

$$\textcircled{1} \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

$$\textcircled{2} \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$\textcircled{3} \alpha (\vec{u} \cdot \vec{v}) = (\alpha \vec{u}) \cdot \vec{v} \\ = \vec{u} \cdot (\alpha \vec{v})$$

Proof of $\textcircled{2}$ when $n=2$:

Let $\vec{u}, \vec{v}, \vec{w}$ be in $\mathbb{R}^2$.

Then, $\vec{u} = \langle x, y \rangle$

$$\vec{v} = \langle p, q \rangle$$

$$\vec{w} = \langle a, b \rangle$$

where $x, y, p, q, a, b \in \mathbb{R}$.

We have that

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \langle x, y \rangle \cdot (\langle p, q \rangle + \langle a, b \rangle)$$

$$= \langle x, y \rangle \cdot \langle p+a, q+b \rangle$$

$$= x(p+a) + y(q+b)$$

$$= xp + xa + yq + yb$$

On the other hand,

$$\vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

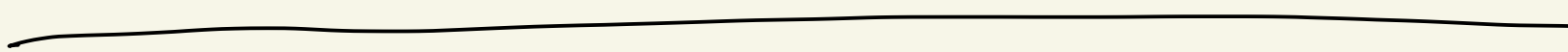
$$= \langle x, y \rangle \cdot \langle p, q \rangle + \langle x, y \rangle \cdot \langle a, b \rangle$$

Q
Q
U
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L

$$= xp + yq + xa + yb \quad \leftarrow$$

Therefore,

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}.$$



Topic 2 - Matrices

Def: A matrix is a rectangular array of numbers.

If M is a matrix and it has m rows and n columns then we say that

M is an $m \times n$ matrix.
read: "m by n"

Abstractly we can write an $m \times n$ matrix like

this:

$$M = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Where a_{ij} is the entry
in the i -th row and
 j -th column.

Ex: $M = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

M is 2×2 .

$$a_{11} = 1$$

$$a_{12} = 2$$

$$a_{21} = -1$$

$$a_{22} = 0$$

Ex: $M = (1 \quad 5 \quad 3 \quad 0)$

||
 $(a_{11} \quad a_{12} \quad a_{13} \quad a_{14})$

M is 1×4 .

$$a_{11} = 1$$

$$a_{12} = 5$$

$$a_{13} = 3$$

$$a_{14} = 0$$

Def: Two matrices M and N are equal if they have the same dimensions and the same entries.

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

Note: Sometimes we want to think of a vector as a matrix.

Suppose $\vec{v} = \langle a_1, a_2, \dots, a_n \rangle$
is in $\mathbb{R}^n$.

We can think of it
as a $1 \times n$ matrix

$$\vec{v} = (a_1 \ a_2 \ \dots \ a_n)$$

or as an $n \times 1$ matrix

$$\vec{v} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Ex: $\vec{v} = \langle 1, 2, 5 \rangle$

Can think of $\vec{v}$ as

$$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \leftarrow 3 \times 1 \text{ matrix}$$

$(1 \ 2 \ 5) \leftarrow 1 \times 3 \text{ matrix}$
