Math 2550
8/30/23

Proof of property ③ when $n=2$

$$\alpha(\beta\vec{u}) = (\alpha\beta)\vec{u}$$

<u>proof:</u> Let $\vec{u}$ be a vector in $\mathbb{R}^2$. Let α, β be in $\mathbb{R}$.

Then, $\vec{u} = \langle x, y \rangle$ where x and y are in $\mathbb{R}$.

So,

$$\begin{aligned}
\alpha(\beta\vec{u}) &= \alpha(\beta\langle x, y\rangle) \\
&= \alpha\langle \beta x, \beta y\rangle \\
&= \langle \alpha\beta x, \alpha\beta y\rangle
\end{aligned}$$

Also,

$$(\alpha\beta)\vec{u} = (\alpha\beta)\langle x, y\rangle$$

$$= \langle \alpha\beta x, \alpha\beta y\rangle$$

the same

Therefore,

$$\alpha(\beta\vec{u}) = (\alpha\beta)\vec{u}.$$

$\blacksquare$

Let's now prove ⑤ when $n=3$. $\left[\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}\right]$

<u>proof:</u> Let α be a real number. Let $\vec{u}, \vec{v}$ be in $\mathbb{R}^3$.

Then, $\vec{u} = \langle x, y, z\rangle$

and $\vec{v} = \langle a, b, c \rangle$

where x, y, z, a, b, c are real #s.

Then,

$$\alpha(\vec{u} + \vec{v})$$

$$= \alpha(\langle x, y, z \rangle + \langle a, b, c \rangle)$$

$$= \alpha \langle x + a, y + b, z + c \rangle$$

$$= \langle \alpha(x+a), \alpha(y+b), \alpha(z+c) \rangle$$

$$= \langle \alpha x + \alpha a, \alpha y + \alpha b, \alpha z + \alpha c \rangle$$

$$= \langle \alpha x, \alpha y, \alpha z \rangle + \langle \alpha a, \alpha b, \alpha c \rangle$$

$$= \alpha \langle x, y, z \rangle + \alpha \langle a, b, c \rangle$$

$= \alpha \vec{u} + \alpha \vec{v}.$ $\blacksquare$

$\square$ $\blacksquare$ QED

(end of proof markers)

Def: Let $\vec{v}$ and $\vec{w}$ be in $\mathbb{R}^n$.

where $\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$

and $\vec{w} = \langle b_1, b_2, \ldots, b_n \rangle$

The dot product of $\vec{v}$ and $\vec{w}$ is defined to be

$$\vec{v} \cdot \vec{w} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Note that $\vec{v} \cdot \vec{w}$ is a number.

<u>Ex:</u> In $\mathbb{R}^2$, let $\vec{v} = \langle 3, 5 \rangle$ and $\vec{w} = \langle 7, -1 \rangle$.

Then,

$$\vec{v} \cdot \vec{w} = \langle 3, 5 \rangle \cdot \langle 7, -1 \rangle$$

$$= (3)(7) + (5)(-1)$$

$$= 21 - 5$$

$$= 16$$

<u>Ex:</u> In $\mathbb{R}^5$, let

$\vec{v} = \langle 1, 1, 7, -5, 2 \rangle$ and

$\vec{w} = \langle 0, \frac{1}{2}, -1, 5, 8 \rangle$.

Then,

$$\vec{v} \cdot \vec{w} = \langle 1, 1, 7, -5, 2 \rangle \cdot \langle 0, \tfrac{1}{2}, -1, 5, 8 \rangle$$

$$= (1)(0) + (1)(\tfrac{1}{2}) + (7)(-1) + (-5)(5) + (2)(8)$$

$$= 0.5 - 7 - 25 + 16$$

$$= -16 + 0.5 = -15.5$$

Properties of the dot product

Let $\vec{u}, \vec{v}, \vec{w}$ be in $\mathbb{R}^n$.
Let α be in $\mathbb{R}$.

Then:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $\alpha(\vec{u} \cdot \vec{v}) = (\alpha \vec{u}) \cdot \vec{v}$
   $= \vec{u} \cdot (\alpha \vec{v})$

Proof of 2 when $n = 2$:

Let $\vec{u}, \vec{v}, \vec{w}$ be in $\mathbb{R}^2$.

Then, $\vec{u} = \langle x, y \rangle$
$\vec{v} = \langle p, q \rangle$
$\vec{w} = \langle a, b \rangle$

where $x, y, p, q, a, b \in \mathbb{R}$.

We have that

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \langle x, y \rangle \cdot \left(\langle p, q \rangle + \langle a, b \rangle \right)$$
$$= \langle x, y \rangle \cdot \langle p+a, q+b \rangle$$
$$= x(p+a) + y(q+b)$$
$$= xp + xa + yq + yb$$

EQUAL

On the other hand,

$\vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
$$= \langle x, y \rangle \cdot \langle p, q \rangle + \langle x, y \rangle \cdot \langle a, b \rangle$$

$= xp + yq + xa + yb$

Therefore,

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}.$$

$\blacksquare$

Topic 2 - Matrices

<u>Def:</u> A <u>matrix</u> is a rectangular array of numbers.
If M is a matrix and it has m rows and n columns then we say that M is an $\underbrace{m \times n}_{\text{read: "m by n"}}$ matrix.

Abstractly we can write an $m \times n$ matrix like this:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where a_{ij} is the entry in the i-th row and j-th column.

Ex: $M = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

M is 2×2.

$a_{11} = 1$

$a_{12} = 2$

$a_{21} = -1$

$a_{22} = 0$

Ex: $M = \begin{pmatrix} 1 & 5 & 3 & 0 \end{pmatrix}$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{pmatrix}$$

M is 1×4.

$a_{11} = 1$
$a_{12} = 5$
$a_{13} = 3$
$a_{14} = 0$

Def: Two matrices M and N are equal if they have the same dimensions and the same entries.

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

Note: Sometimes we want to think of a vectr as a matrix.

Suppose $\vec{v} = \langle a_1, a_2, \dots, a_n \rangle$ is in $\mathbb{R}^n$.

We can think of it as a $1 \times n$ matrix

$$\vec{v} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

or as an $n \times 1$ matrix

$$\vec{v} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Ex: $\vec{v} = \langle 1, 2, 5 \rangle$

Can think of $\vec{v}$ as

$$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \leftarrow 3 \times 1 \text{ matrix}$$

$\begin{pmatrix} 1 & 2 & 5 \end{pmatrix} \leftarrow$ 1x3 matrix