Math 2550-03
4/9/24

Test 2 is in two weeks. Tuesday 4/23.

Review day is Thursday 4/18.

Special example:

The "trivial" vector space is $V = \{\vec{0}\}$ ← the vector space with just 1 vector in it

It turns out there is no basis for V. We define V to have dimension 0.

Ex: Let $V = \mathbb{R}^n$ and $F = \mathbb{R}$.

The standard basis for $\mathbb{R}^n$ is $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ where $\vec{v}_i$ has a 1 in spot i and 0's everywhere else.

This gives $\dim(\mathbb{R}^n) = n$.

n	standard basis for $\mathbb{R}^n$	$\dim(\mathbb{R}^n)$
2	$\vec{v}_1 = \langle 1, 0 \rangle$ $\vec{v}_2 = \langle 0, 1 \rangle$	2
3	$\vec{v}_1 = \langle 1, 0, 0 \rangle$ $\vec{v}_2 = \langle 0, 1, 0 \rangle$ $\vec{v}_3 = \langle 0, 0, 1 \rangle$	3

4	$\vec{v}_1 = \langle 1, 0, 0, 0 \rangle$ $\vec{v}_2 = \langle 0, 1, 0, 0 \rangle$ $\vec{v}_3 = \langle 0, 0, 1, 0 \rangle$ $\vec{v}_4 = \langle 0, 0, 0, 1 \rangle$	4

$\boxed{\text{Ex:}}$

$$P_n = \left\{ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \;\middle|\; \begin{array}{l} a_0, a_1, \ldots, a_n \\ \text{are in } \mathbb{R} \end{array} \right\}$$

polynomials of degree $\leq n$

The standard basis for P_n is

$$\left.\begin{array}{l} \vec{v}_0 = 1 \\ \vec{v}_1 = x \\ \vec{v}_2 = x^2 \\ \vdots \\ \vec{v}_n = x^n \end{array}\right\} \begin{array}{l} n+1 \\ \text{vectors} \end{array}$$

So, $\dim(P_n) = n+1$

n	standard basis for P_n	$\dim(P_n)$
0	1	1
1	$1, x$	2
2	$1, x, x^2$	3
3	$1, x, x^2, x^3$	4
$\vdots$	$\vdots$	$\vdots$

Ex: Let $V = M_{2,2}$ $\leftarrow$ set of 2×2 matrices
and $F = \mathbb{R}$.

Idea:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

One can show [HW 7 - Part I] that

$$\vec{v}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are a basis for $V = M_{2,2}$.

Thus, $\dim(M_{2,2}) = 4$

(The not-a-basis theorem)

<u>Theorem</u>: Let V be a finite-dimensional vector space over a field F with $\dim(V) = n$.

<u>means</u>: V has a basis with n vectors in it

Let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ be vectors from V.

① If $m < n$, then $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ do not span V.

② If $m > n$, then $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ are linearly dependent.

Ex: Let $V = \mathbb{R}^3$, $F = \mathbb{R}$.

Know: $\dim(\mathbb{R}^3) = 3 \leftarrow \boxed{n=3}$

Let

$$\left.\begin{array}{l} \vec{w}_1 = \langle 1, 1, 0 \rangle \\ \vec{w}_2 = \langle 3, 2, 0 \rangle \end{array}\right\} \begin{array}{l} m = 2 \\ \text{vectors} \end{array}$$

We have 2 vectors in a 3-dim space. Since $2 < 3$, the vectors $\vec{w}_1, \vec{w}_2$ will not span $\mathbb{R}^3$.

Ex: Let $V = P_2$, $F = \mathbb{R}$.

We know $\underbrace{\dim(P_2) = 3}_{\text{basis is } 1,\, x,\, x^2}$

Let

$$\left.\begin{aligned} \vec{w}_1 &= 1 + x \\ \vec{w}_2 &= 1 - x^2 \\ \vec{w}_3 &= x \\ \vec{w}_4 &= x^2 \end{aligned}\right\} \text{4 vectors}$$

By part 2 of the above theorem, since we have 4 vectors in a 3-dimensional space and $4 > 3$, the vectors must be linearly dependent.

We can see it this way:

$$1\cdot(1+x) - 1\cdot(1-x^2) - 1\cdot(x) - 1\cdot(x^2) = \vec{0}$$

$$1\cdot\vec{w}_1 - 1\cdot\vec{w}_2 - 1\cdot\vec{w}_3 - 1\cdot\vec{w}_4 = \vec{0}$$

$$\vec{w}_1 = \vec{w}_2 + \vec{w}_3 + \vec{w}_4$$

<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F, with $n = \dim(V)$.

[So, V has a basis with n vectors in it.]

Suppose $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are vectors from V.

① If $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are linearly independent, then $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ must span V and hence are a basis for V.

② If $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ span V, then $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are linearly independent and hence are a basis for V.

Ex: Let $V = P_2$, $F = \mathbb{R}$.

We saw last time that $1, x, x^2$ is a basis for P_2.

Thus, $\dim(P_2) = 3$.

Consider

$$\left.\begin{aligned} \vec{w}_1 &= 1 \\ \vec{w}_2 &= 1+x \\ \vec{w}_3 &= 1+x+x^2 \end{aligned}\right\} \text{3 vectors}$$

Previously we showed these 3 vectors $1, 1+x, 1+x+x^2$ are linearly independent.

By the theorem, since we

have 3 lin. ind. vectors in a 3-dim space, they are a basis for P_2.

So, $1, 1+x, 1+x+x^2$ is also a basis for P_2.