Math 2550-03

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Topic 8 – Eigenvalues and Eigenvectors

<u>Def:</u> Let A be an $n \times n$ matrix. Suppose $\vec{x}$ in $\mathbb{R}^n$ and $\vec{x} \neq \vec{0}$ and $\boxed{A\vec{x} = \lambda \vec{x}}$ for some scalar/number λ.

λ is lambda

Then λ is called an <u>eigenvalue</u> of A and $\vec{x}$ is called an <u>eigenvector</u> of A corresponding to λ. Given an eigenvalue λ of A, the <u>eigenspace</u> of A corresponding to λ is

$$E_\lambda(A) = \{ \vec{x} \mid A\vec{x} = \lambda \vec{x} \}$$

[$E_\lambda(A)$ consists of all eigenvectors corresponding to λ and also the zero vector $\vec{0}$]

Ex: Let $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Let $\vec{x} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$.

Then,

$$A\vec{x} = \underbrace{\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}}_{3\times 3} \underbrace{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}_{3\times 1}$$

$3 \times 3 \quad \checkmark \quad 3 \times 1$

answer $= 3 \times 1$

$$= \begin{pmatrix} (0)(-2) + (0)(1) + (-2)(1) \\ (1)(-2) + (2)(1) + (1)(1) \\ (1)(-2) + (0)(1) + (3)(1) \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot \vec{x}$$

So, $A\vec{x} = 1 \cdot \vec{x}$

$A\vec{x} = \lambda \vec{x}$

$\lambda = 1$

So, $\lambda = 1$ is an eigenvalue of A and $\vec{x} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 1$.

How do we find the eigenvalues of an $n \times n$ matrix A?

Suppose λ is an eigenvalue of A and $\vec{x} \neq \vec{0}$ is an eigenvector associated with λ.

Then, $A\vec{x} = \lambda \vec{x}$.

So, $A\vec{x} - \lambda \vec{x} = \vec{0}$.

using $I_n \vec{x} = \vec{x}$

Then, $(A - \lambda I_n)\vec{x} = \vec{0}$ where I_n is the $n \times n$ identity matrix.

So, $(A-\lambda I_n)\vec{x}=\vec{0}$ where $\vec{x}\neq\vec{0}$.

The only way this can happen is if $A-\lambda I_n$ has no inverse.

Why? Let $B=A-\lambda I_n$.

If B^{-1} existed then since $B\vec{x}=\vec{0}$ you would get $B^{-1}B\vec{x}=B^{-1}\vec{0}$ which would give $\vec{x}=\vec{0}$. But $\vec{x}\neq\vec{0}$. So, B^{-1} does not exist

Thus, $\det(A-\lambda I_n)=0$ since $(A-\lambda I_n)^{-1}$ does not exist.

Summary: The eigenvalues of A satisfy the equation

$$\underbrace{\det(A - \lambda I_n)}_{\text{called the characteristic polynomial of } A} = 0.$$

Ex: Let $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Let's find the eigenvalues of A. We need to solve

$$\det(A - \lambda I_3) = 0$$

because A is 3×3

We have

$$\det(A - \lambda I_3)$$

$$= \det\left(\underbrace{\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}}_{A} - \lambda \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3} \right)$$

$$= \det\left(\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$$

expand on column 2

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= -0 + (2-\lambda)\underbrace{\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix}} - 0$$

$$\begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$$

$$= (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(-\lambda)(3-\lambda) - (1)(-2)\right]$$

$$= (2-\lambda)(\lambda^2 - 3\lambda + 2)$$

we will use this below

$$= 2\lambda^2 - 6\lambda + 4 - \lambda^3 + 3\lambda^2 - 2\lambda$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

characteristic polynomial of A

The eigenvalues of A are the λ that solve

$$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

From above this becomes

$$(2-\lambda)(\lambda^2 - 3\lambda + 2) = 0$$

which becomes

$$\underbrace{(2-\lambda)}_{\text{factor out } (-1)}(\lambda-1)(\lambda-2) = 0$$

$$-(\lambda-2)(\lambda-1)(\lambda-2) = 0$$

So we get

$$-(\lambda-2)^2(\lambda-1) = 0$$

The eigenvalues/roots are $\lambda = 1, 2$

Let's find the eigenvectors of A.

Let's start with $\lambda = 1$.

Let's find a basis for

$$E_1(A) = \{ \vec{x} \mid A\vec{x} = 1 \cdot \vec{x} \}$$

The equation $A\vec{x} = 1 \cdot \vec{x}$ becomes

$$\underbrace{\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{A\vec{x}} = \underbrace{1 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{1 \cdot \vec{x}}$$

This becomes

$$\begin{pmatrix} 0a + 0b - 2c \\ a + 2b + c \\ a + 0b + 3c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This gives $\begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

This gives $\begin{pmatrix} -a-2c \\ a+b+c \\ a+2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

So, $\boxed{\begin{aligned} -a \quad -2c &= 0 \\ a+b+c &= 0 \\ a \quad +2c &= 0 \end{aligned}}$

Solving we get

$$\left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \xrightarrow{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We get

$$\boxed{\begin{aligned} a \quad\quad + 2c &= 0 \quad (1) \\ b - c &= 0 \quad (2) \\ 0 &= 0 \quad (3) \end{aligned}}$$

leading: a, b
free: c

Solving:

$$\boxed{\begin{aligned} &c = t \\ (2)\ &b = c = t \\ (1)\ &a = -2c = -2t \end{aligned}}$$

Thus, if $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is in $E_1(A)$

then $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

So, $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is a basis for $E_1(A)$

Thus, $\dim(E_1(A)) = 1$

Ex: When $t=1$ we get $\vec{x} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ that we used earlier.