Math 2550-03
4/11/24

What's a basis good for?
To make a coordinate system.

<u>Theorem:</u> Let V be a vector space over a field F.
Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a basis for V. Then given any vector $\vec{v}$ from V there exist unique scalars $c_1, c_2, \ldots, c_n$ where

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n$$

Ex: $V=\mathbb{R}^2$, $F=\mathbb{R}$

Previously we saw that

$\vec{v}_1=\langle 2,1\rangle$, $\vec{v}_2=\langle -1,1\rangle$

is a basis for $\mathbb{R}^2$.

Also we saw that for any $\vec{v}=\langle a,b\rangle$ we had

$$\underbrace{\langle a,b\rangle = \left(\tfrac{1}{3}a+\tfrac{1}{3}b\right)\langle 2,1\rangle + \left(-\tfrac{1}{3}a+\tfrac{2}{3}b\right)\langle -1,1\rangle}_{\vec{v}=c_1\vec{v}_1+c_2\vec{v}_2}$$

Ex:

$$\boxed{\langle 3,-6\rangle = (-1)\cdot\langle 2,1\rangle + (-5)\cdot\langle -1,1\rangle}$$

$\boxed{a=3,\ b=-6}$

If instead you used the standard basis $\vec{w}_1 = \langle 1, 0 \rangle$, $\vec{w}_2 = \langle 0, 1 \rangle$ then you'd have

$$\langle 3, -6 \rangle = 3 \cdot \langle 1, 0 \rangle + (-6) \cdot \langle 0, 1 \rangle$$

<u>Def:</u> Let V be a vector space over a field F. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a basis for V.
If we fix this ordering on the basis elements, then we call this an <u>ordered basis</u> for V.

We write

$$\beta = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$$

name of basis

β = beta

brackets mean that the order matters

to denote an ordered basis.

Given any vector $\vec{v}$ from V

we can write

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

The constants $c_1, c_2, \ldots, c_n$ are called the <u>coordinates of $\vec{v}$ with respect to the basis β.</u>

We write

$$\underbrace{[\vec{v}]_\beta = \langle c_1, c_2, \ldots, c_n \rangle}_{\text{coordinate vector of } \vec{v} \text{ w/ respect to } \beta}$$

can write

$$[\vec{v}]_\beta = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

<u>Ex:</u> $V = \mathbb{R}^2$, $F = \mathbb{R}$

Let

$$\left.\begin{aligned} \beta &= [\langle 1,0\rangle, \langle 0,1\rangle] \\ \beta' &= [\langle 0,1\rangle, \langle 1,0\rangle] \end{aligned}\right\} \text{two different orderings on the standard basis}$$

$$\left.\begin{aligned} \gamma &= [\langle 2,1\rangle, \langle -1,1\rangle] \\ \gamma' &= [\langle -1,1\rangle, \langle 2,1\rangle] \end{aligned}\right\} \text{two different orderings of basis } \langle 2,1\rangle, \langle -1,1\rangle$$

Let $\vec{v} = \langle 3,-6\rangle$.

Note

$$\vec{v} = \langle 3, -6 \rangle = 3 \cdot \langle 1, 0 \rangle + (-6) \cdot \langle 0, 1 \rangle$$

$$\boxed{\beta = [\langle 1, 0 \rangle, \langle 0, 1 \rangle]}$$

So,

$$[\vec{v}]_{\beta} = \langle 3, -6 \rangle$$

Note

$$\vec{v} = \langle 3, -6 \rangle = (-6) \langle 0, 1 \rangle + (3) \langle 1, 0 \rangle$$

$$\boxed{\beta' = [\langle 0, 1 \rangle, \langle 1, 0 \rangle]}$$

$$[\vec{v}]_{\beta'} = \langle -6, 3 \rangle$$

$$\boxed{\gamma = [\langle 2, 1 \rangle, \langle -1, 1 \rangle]}$$

Note

$$\vec{v} = \langle 3, -6 \rangle = (-1) \langle 2, 1 \rangle + (-5) \langle -1, 1 \rangle$$

So, $[\vec{v}]_\gamma = \langle -1, -5 \rangle$

$\gamma' = [\langle -1, 1\rangle, \langle 2, 1\rangle]$

Note,

$\vec{v} = \langle 3, -6 \rangle = (-5)\langle -1, 1\rangle + (-1)\langle 2, 1\rangle$

So,

$[\vec{v}]_{\gamma'} = \langle -5, -1 \rangle$

<u>Q</u>: What if you know that

$[\vec{w}]_\gamma = \langle 3, -2 \rangle$.

$\gamma = [\langle 2, 1\rangle, \langle -1, 1\rangle]$

What is $\vec{w}$?

Then,

$$\vec{w} = 3\langle 2,1\rangle + (-2)\langle -1,1\rangle$$
$$= \langle 6,3\rangle + \langle 2,-2\rangle$$
$$= \langle 8,1\rangle$$

<u>Ex:</u> $V = P_2$, $F = \mathbb{R}$

polys of degree ≤ 2

Let

$\beta = [1, x, x^2]$ ← standard basis

$\gamma = [1, 1+x, 1+x+x^2]$ ← other basis we found

Let $\vec{v} = 4 + 2x + 3x^2$

Find $[\vec{v}]_\beta$ and $[\vec{v}]_\gamma$.

Note

$$\vec{v} = 4 \cdot 1 + 2 \cdot x + 3 \cdot x^2$$

$$\boxed{\beta = [1, x, x^2]}$$

So,

$$[\vec{v}]_\beta = \langle 4, 2, 3 \rangle$$

To find $[\vec{v}]_\gamma$ we must solve

$$\underbrace{4 + 2x + 3x^2}_{\vec{v}} = c_1(1) + c_2(1+x) + c_3(1+x+x^2)$$

$$\boxed{\gamma = [1, 1+x, 1+x+x^2]}$$

This becomes

$$4 + 2x + 3x^2 = c_1 + c_2 + c_2 x + c_3 + c_3 x + c_3 x^2$$

which gives

$$4+2x+3x^2=(c_1+c_2+c_3)+(c_2+c_3)x+c_3x^2$$

This gives

$$\boxed{\begin{aligned} c_1+c_2+c_3&=4 \quad (1)\\ c_2+c_3&=2 \quad (2)\\ c_3&=3 \quad (3)\end{aligned}} \Rightarrow \boxed{\begin{aligned} (3)\ c_3&=3\\ (2)\ c_2&=2-c_3\\ &=2-3=-1\\ (1)\ c_1&=4-c_2-c_3\\ &=4+1-3\\ &=2\end{aligned}}$$

Thus,

$$4+2x+3x^2=2(1)+(-1)(1+x)+(3)(1+x+x^2)$$

$$\boxed{\gamma=[1,1+x,1+x+x^2]}$$

So,

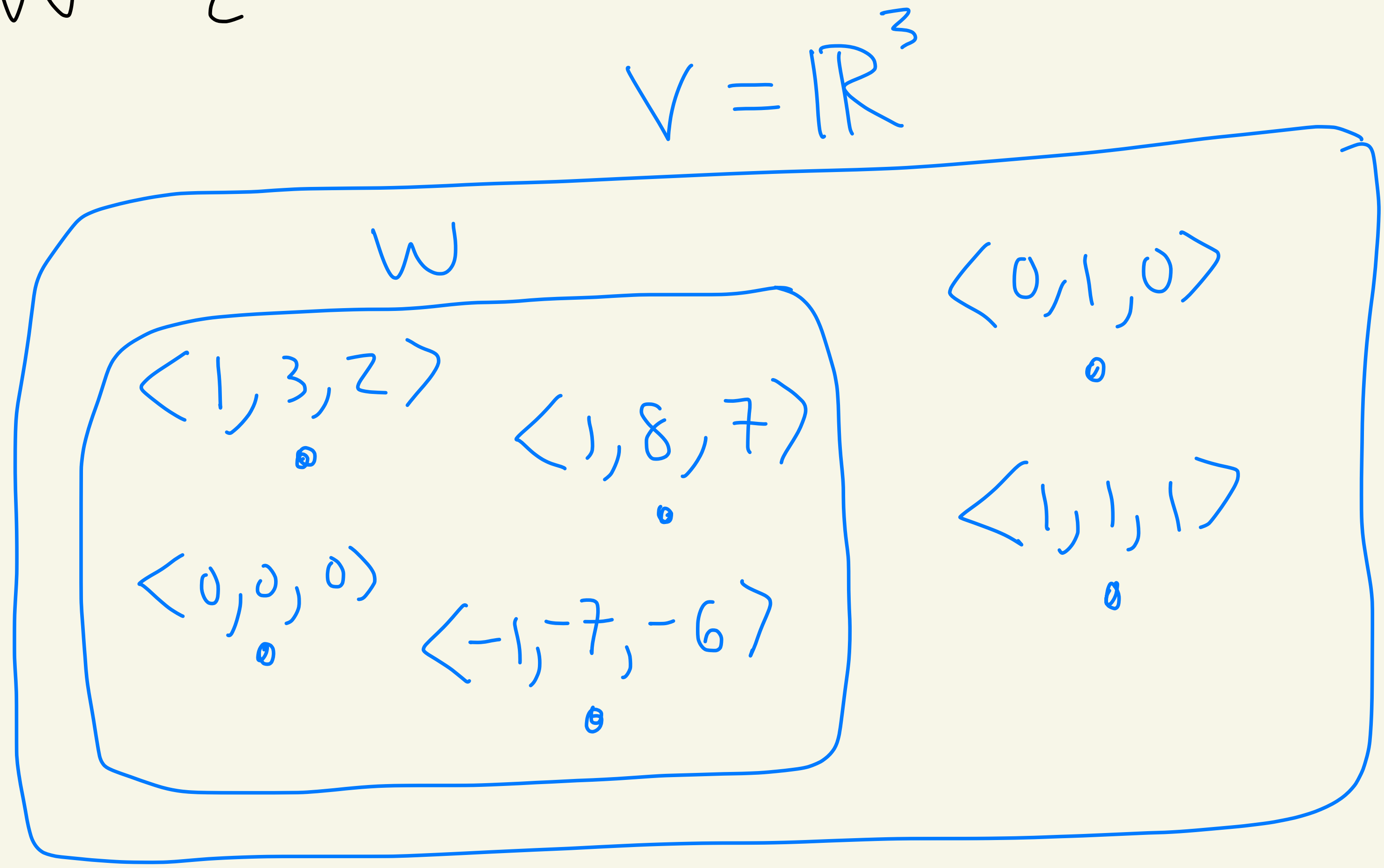
$$[4+2x+3x^2]_\gamma=\langle 2,-1,3\rangle$$

Let's do an example where we find a basis for a subspace!

$\boxed{\text{HW 7 - Part 2}}$

1 (b) $V = \mathbb{R}^3$, $F = \mathbb{R}$

$$W = \{ \langle a, b, c \rangle \mid b = a + c,\ a, b, c \in \mathbb{R} \}$$



In HW you show that W is a subspace for $\mathbb{R}^3$.

Let's find a basis for W.

Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be in W.

Then, $b = a + c$.

So,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ a+c \\ c \end{pmatrix} = \begin{pmatrix} a \\ a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c \\ c \end{pmatrix}$$

$$= a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Thus, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ span W.

[Ex: $\begin{pmatrix} 5 \\ -1 \\ -6 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 6\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$]

Let's show $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$ are lin. ind.

We must solve

$$c_1\begin{pmatrix}1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

for c_1, c_2.

This becomes $\begin{pmatrix}c_1\\c_1+c_2\\c_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ $\begin{matrix}\leftarrow c_1 = 0\\ \leftarrow c_1+c_2=0\\ \leftarrow c_2=0\end{matrix}$

So, $c_1 = 0, c_2 = 0$ is the only solution.

Thus, $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$ are lin. ind.

Therefore $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$ are a basis for W. Thus, $\dim(W) = 2$.

So, W is a 2-dimensional space inside the 3-dimensional space $V = \mathbb{R}^3$.