Math 2550-03
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3/5 Finish determinants	3/7 Review
3/12 Test 1	3/14 topic 6 starts

Last time we showed that

$$\det\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} = -1$$

by expanding on column 2.

Let's compute this again but expand on a row.

Ex:

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} =$$

$$(3)\cdot\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (1)\cdot\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + (0)\cdot\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \qquad \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \qquad \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= 3\cdot\left[(-4)(-2) - (3)(4)\right]$$

$$-1\cdot\left[(-2)(-2)-(3)(5)\right] + 0$$

$$= 3\cdot[8-12] - [4-15] = 3(-4) - (-11)$$

$$= -12 + 11 = -1$$

Ex:

$$\det\begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

expand on row 1

$$= (1)\cdot\begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & -2 \\ 1 & 2 & 0 \end{vmatrix} \longleftarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$$-(0)\cdot\begin{vmatrix} 2 & 3 & 0 \\ 1 & 0 & -2 \\ 1 & 2 & 0 \end{vmatrix} \longleftarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$$+(0)\cdot\begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & -2 \\ 1 & 1 & 0 \end{vmatrix} \longleftarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$$-(-1)\cdot\begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{vmatrix} \longleftarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$$= \underbrace{\begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & -2 \\ 1 & 2 & 0 \end{vmatrix}} + \underbrace{\begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{vmatrix}}$$

$$= -0 + 0 - (-2)\cdot \underbrace{\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}}_{\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}}$$

$$- 1 \cdot \underbrace{\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}}_{\begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}} + 0 - 0$$

$$\boxed{\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}}$$

$$= 2\cdot \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= 2\cdot \left[(1)(2) - (1)(3)\right] - \left[(1)(2) - (1)(3)\right]$$

$$= 2\cdot[-1] - [-1] = -1$$

Properties of the determinant

Let A and B be $n \times n$ matrices.

① $\det(A^T) = \det(A)$

② $\det(AB) = \det(A) \cdot \det(B)$

③ If A^{-1} exists, then $\det(A) \neq 0$

④ If $\det(A) \neq 0$, then A^{-1} exists.

[So, if $\det(A) = 0$, then A^{-1} will <u>not</u> exist]

⑤ If A^{-1} exists, then $\det(A^{-1}) = \frac{1}{\det(A)}$

⑥ $\det(I_n) = 1$ where I_n is the $n \times n$ identity matrix

7 If A has a row of zeros or a column of zeros, then $\det(A) = 0$

ex:

$$\det\begin{pmatrix} 5 & 3 & 0 \\ 2 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = 0$$

column of zeros

8 If a row is a multiple of another row in A, or a column is a multiple of another column in A, then $\det(A) = 0$

ex:

$$\det\begin{pmatrix} 1 & 3 & 5 \\ 0 & 0 & 2 \\ -1 & -3 & 7 \end{pmatrix} = 0$$

(column 2)
$= 3 \cdot$ (column 1)

Formula for A^{-1} for 2×2 matrices

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then, $\det(A) = ad - bc$

If $\det(A) \neq 0$, then A^{-1} exists and

$$A^{-1} = \underbrace{\frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{\frac{1}{\det(A)} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

$\boxed{\text{Ex:}}$ Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad$ A^{-1} exists

$\det(A) = (1)(4) - (2)(3) = -2 \neq 0$

$$A^{-1} = \frac{1}{-2} \cdot \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

In general, if A is $n \times n$ and $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \cdot M$$

where M is the adjugate matrix.

wiki has an article about it