Math 2550-03
3/28/24

We are going to define a basis which is a way to create a coordinate system in a vector space.

Def: Let V be a vector space over a field F. Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be vectors from V. We say that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ form a basis for V if:

1. $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span V
2. $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent.

(1) guarantees that every vector $\vec{v}$ from V can be written as
$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$
(2) guarantees that there are no redundencies in $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

Let $\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$

(1) We showed previously that $\vec{v}_1, \vec{v}_2$ span $V = \mathbb{R}^2$.

This is because any vector $\vec{v} = \langle a, b \rangle$ can be written as

$$\langle a, b \rangle = a\langle 1, 0 \rangle + b\langle 0, 1 \rangle = a\vec{v}_1 + b\vec{v}_2$$

ex: $\langle 5,-1\rangle = 5\langle 1,0\rangle - 1\cdot\langle 0,1\rangle$
$= 5\vec{v}_1 - 1\cdot\vec{v}_2$

(2) On Tuesday, we showed that $\vec{v}_1 = \langle 1,0\rangle$, $\vec{v}_2 = \langle 0,1\rangle$ are linearly independent.

By (1) & (2), $\vec{v}_1 = \langle 1,0\rangle$, $\vec{v}_2 = \langle 0,1\rangle$ form a basis for $V = \mathbb{R}^2$.

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.
Let $\vec{v}_1 = \langle 2,1\rangle$, $\vec{v}_2 = \langle -1,1\rangle$

(1) Previously we showed that $\vec{v}_1, \vec{v}_2$ span $V = \mathbb{R}^2$, in fact

We showed that any vector $\langle a, b \rangle$ can be written as

$$\langle a,b\rangle = \left(\tfrac{1}{3}a+\tfrac{1}{3}b\right)\cdot\langle 2,1\rangle + \left(-\tfrac{1}{3}a+\tfrac{2}{3}b\right)\cdot\langle -1,1\rangle$$
$$= \left(\tfrac{1}{3}a+\tfrac{1}{3}b\right)\cdot\vec{v}_1 + \left(-\tfrac{1}{3}a+\tfrac{2}{3}b\right)\cdot\vec{v}_2$$

Ex:

$$\langle 3,6\rangle = 3\cdot\langle 2,1\rangle + 3\cdot\langle -1,1\rangle$$
$$= 3\cdot\vec{v}_1 + 3\cdot\vec{v}_2$$

(2) We never checked linear independence.

We need to solve

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$$

for c_1, c_2.

This becomes

$$c_1\langle 2,1\rangle + c_2\langle -1,1\rangle = \langle 0,0\rangle$$

which is

$$\langle 2c_1 - c_2,\ c_1 + c_2\rangle = \langle 0, 0\rangle$$

This gives

$$\boxed{\begin{aligned} 2c_1 - c_2 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}}$$

Solving time!

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{-2R_1 + R_2 \rightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -3 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 &= 0 \\ c_2 &= 0 \end{aligned}} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

So,

(2) $c_2 = 0$

(1) $c_1 = -c_2 = -(0) = 0.$

Thus, the only solution to

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$$

is $c_1 = 0$, $c_2 = 0$.

So, $\vec{v}_1 = \langle 2, 1 \rangle$, $\vec{v}_2 = \langle -1, 1 \rangle$

are linearly independent.

By the above, we know that $\vec{v}_1 = \langle 2, 1 \rangle$, $\vec{v}_2 = \langle -1, 1 \rangle$ are a basis for $V = \mathbb{R}^2$.

<u>Theorem:</u> Let V be a vector space over a field F. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a basis with n vectors for V. Then any other basis for V will also have n vectors in it.

All bases for V have the same number of vectors in them

<u>Ex:</u> $V = \mathbb{R}^2$, $F = \mathbb{R}$

<u>basis #1</u>: $\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$

<u>basis #2</u>: $\vec{v}_1 = \langle 2, 1 \rangle$, $\vec{v}_2 = \langle -1, 1 \rangle$

Both bases have $n = 2$ vectors.
By the theorem, any basis for $V = \mathbb{R}^2$ will have 2 vectors in it.

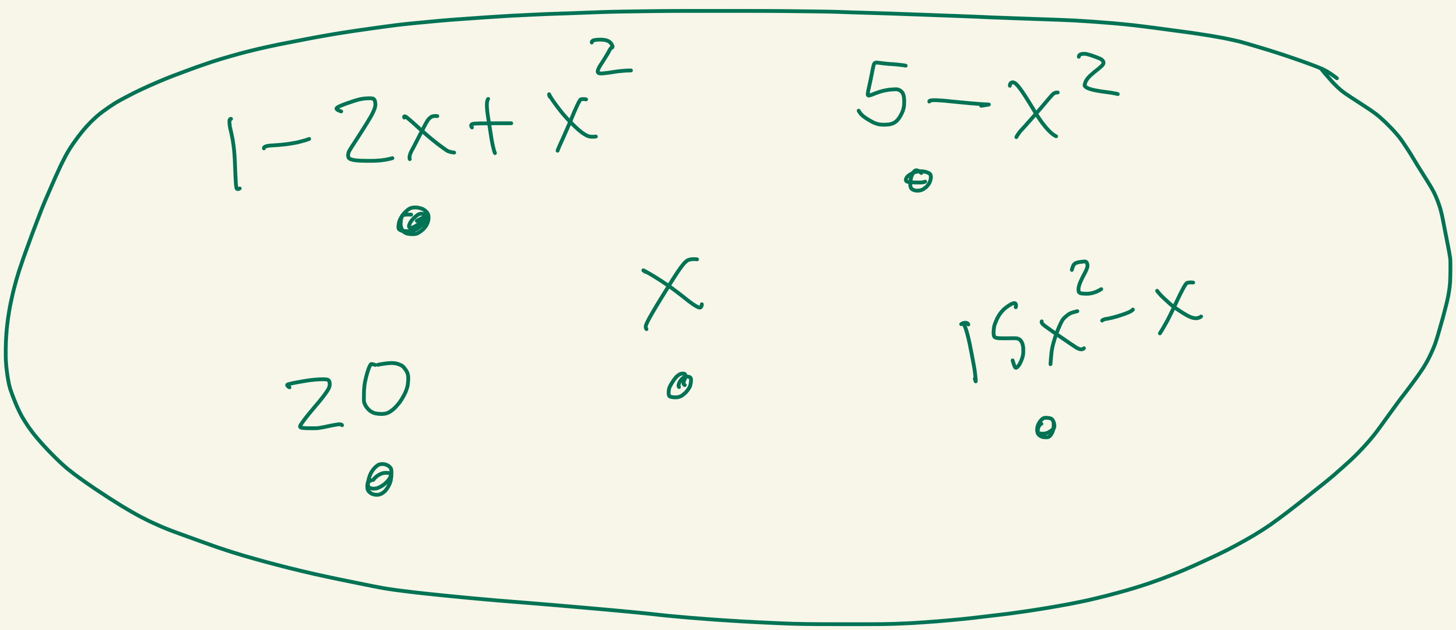
Def: Let V be a vector space over a field F. If there exists a basis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ with n vectors for V, then we say that V is finite-dimensional a has dimension n and we write $\dim(V) = n$.

Ex: $\dim(\mathbb{R}^2) = 2$ because $\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$ is a basis with $n = 2$ vectors.

Ex: $V = P_2$ ← all polynomials of degree ≤ 2

$F = \mathbb{R}$

$V = P_2$



Let $\vec{v}_1 = 1$, $\vec{v}_2 = x$, $\vec{v}_3 = x^2$.

$$\boxed{\begin{array}{l}\underline{\text{Claim:}}\ \vec{v}_1 = 1,\ \vec{v}_2 = x,\ \vec{v}_3 = x^2 \\ \text{is a basis for } V = P_2\end{array}}$$

pf of claim:

1 (spanning) Given a vector we can write $\vec{v} = a + bx + cx^2$

$$\vec{v} = a \cdot 1 + b \cdot x + c \cdot x^2$$
$$= a \cdot \vec{v}_1 + b \cdot \vec{v}_2 + c \cdot \vec{v}_3$$

So, $\vec{v}_1 = 1$, $\vec{v}_2 = x$, $\vec{v}_3 = x^2$ span $V = P_2$

$$\boxed{\begin{array}{l}\underline{\text{Ex:}} \\ 5 - x + 3x^2 = 5 \cdot \vec{v}_1 - 1 \cdot \vec{v}_2 + 3 \cdot \vec{v}_3\end{array}}$$

2 (linear independence)

We need to solve

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

for c_1, c_2, c_3.

This becomes

$$c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 = 0 + 0x + 0x^2$$

which gives $c_1 = 0, c_2 = 0, c_3 = 0$.

The only solution to

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

is $c_1 = 0, c_2 = 0, c_3 = 0$

which says that

$\vec{v}_1 = 1, \vec{v}_2 = x, \vec{v}_3 = x^2$

are linearly independent.

By 1 and 2 we have that $\vec{v}_1 = 1$, $\vec{v}_2 = x$, $\vec{v}_3 = x^2$ are a basis for $V = P_2$

$\boxed{\text{Claim}}$

<u>Note:</u> $\dim(P_2) = 3$

since $\vec{v}_1 = 1$, $\vec{v}_2 = x$, $\vec{v}_3 = x^2$ is a basis for P_2 with 3 vectors.