Math 2550-03
3/26/24

Def: Let V be a vector space over a field F.

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be vectors in V.

We say that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are **linearly dependent** if there exist scalars/numbers $c_1, c_2, \ldots, c_n$ from F, that are not all equal to 0 (but some can be zero) where

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{0}.$$

If no such numbers exist then the vectors are called **linearly independent.**

Reformulated:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$$

always has at least one solution, which is $c_1 = 0, c_2 = 0, \dots, c_n = 0$.

If thats the only solution, the vectors are linearly independent.

If theres more solutions then the vectors are linearly dependent.

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

Let $\vec{v}_1 = \langle 3, -5 \rangle$, $\vec{v}_2 = \langle -6, 10 \rangle$

Some solutions to

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$$

are:

$$\underbrace{0 \cdot \langle 3, -5 \rangle + 0 \cdot \langle -6, 10 \rangle = \langle 0, 0 \rangle}_{0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 = \vec{0} \text{ with } c_1 = 0,\ c_2 = 0}$$

$$\underbrace{1\cdot\langle 3,-5\rangle+\tfrac{1}{2}\cdot\langle -6,10\rangle=\langle 0,0\rangle}_{1\cdot\vec{v}_1+\frac{1}{2}\cdot\vec{v}_2=\vec{0}}$$

$c_1=1,\ c_2=1/2$

The vectors $\vec{v}_1=\langle 3,-5\rangle$, $\vec{v}_2=\langle -6,10\rangle$ are linearly dependent because of $1\cdot\vec{v}_1+\frac{1}{2}\cdot\vec{v}_2=\vec{0}$.

<u>Note</u>: From $1\cdot\vec{v}_1+\frac{1}{2}\vec{v}_2=\vec{0}$ we can get

$\vec{v}_1=-\frac{1}{2}\vec{v}_2$ ← $\vec{v}_1$ is a linear combo of $\vec{v}_2$

$\vec{v}_2=-2\vec{v}_1$ ← $\vec{v}_2$ is a linear combo of $\vec{v}_1$

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

Let $\vec{v}_1 = \langle 1, -1 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$, $\vec{v}_3 = \langle 2, -1 \rangle$.

Then,

$$\underbrace{\underbrace{2 \cdot \langle 1, -1 \rangle + 1 \cdot \langle 0, 1 \rangle - 1 \cdot \langle 2, -1 \rangle = \langle 0, 0 \rangle}_{2 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 - 1 \cdot \vec{v}_3 = \vec{0}}}_{c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}}$$

with $c_1 = 2$, $c_2 = 1$, $c_3 = -1$

So, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

Note: $\vec{v}_1 = -\frac{1}{2} \vec{v}_2 + \frac{1}{2} \vec{v}_3$

$\vec{v}_1$ is a linear combination of $\vec{v}_2$ & $\vec{v}_3$

$$\left.\begin{aligned} \vec{v}_2 &= -2 \vec{v}_1 + 1 \cdot \vec{v}_3 \\ \vec{v}_3 &= 2 \vec{v}_1 + 1 \cdot \vec{v}_2 \end{aligned}\right\} \text{other linear combos}$$

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$

$\vec{v}_1 = \langle 1, 1 \rangle$, $\vec{v}_2 = \langle 6, 7 \rangle$, $\vec{v}_3 = \langle -2, -2 \rangle$

These vectors are linearly dependent since

$$\underbrace{2 \cdot \langle 1, 1 \rangle + 0 \cdot \langle 6, 7 \rangle + 1 \cdot \langle -2, -2 \rangle = \langle 0, 0 \rangle}_{2 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 1 \cdot \vec{v}_3 = \vec{0}}$$

$c_1 = 2$, $c_2 = 0$, $c_3 = 1$
are not all zero

Note: $\vec{v}_3 = -2\vec{v}_1 + 0\vec{v}_2$

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$

Let $\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$.

Are $\vec{v}_1, \vec{v}_2$ lin. dep. or lin. ind.?

We need to solve

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$$

which becomes

$$c_1 \langle 1, 0 \rangle + c_2 \langle 0, 1 \rangle = \langle 0, 0 \rangle$$

which becomes

$$\langle c_1, 0 \rangle + \langle 0, c_2 \rangle = \langle 0, 0 \rangle$$

which gives

$$\langle c_1, c_2 \rangle = \langle 0, 0 \rangle$$

So, $c_1 = 0$, $c_2 = 0$.

Thus the only sol. to $c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$ is $c_1 = 0, c_2 = 0$. So, $\vec{v}_1, \vec{v}_2$ are linearly independent.

<u>Ex:</u> Let $V = \mathbb{R}^3$, $F = \mathbb{R}$.

Let $\vec{v}_1 = \langle 1,1,1 \rangle$, $\vec{v}_2 = \langle 1,0,1 \rangle$,

$\vec{v}_3 = \langle 1, \frac{4}{3}, 1 \rangle$.

Are these vectors lin. ind. or lin. dep.?

We want the solutions to

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

which is

$$c_1 \langle 1,1,1 \rangle + c_2 \langle 1,0,1 \rangle + c_3 \langle 1, \tfrac{4}{3}, 1 \rangle = \langle 0,0,0 \rangle$$

which becomes

$$\langle c_1, c_1, c_1 \rangle + \langle c_2, 0, c_2 \rangle + \langle c_3, \tfrac{4}{3} c_3, c_3 \rangle = \langle 0,0,0 \rangle$$

which gives

$$\langle c_1 + c_2 + c_3, c_1 + \tfrac{4}{3} c_3, c_1 + c_2 + c_3 \rangle = \langle 0,0,0 \rangle$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 \qquad + \tfrac{4}{3} c_3 &= 0 \\ c_1 + c_2 + c_3 &= 0 \end{aligned}}$$

Solving:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & 4/3 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow[-R_1 + R_3 \to R_3]{-R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 0 \quad (1) \\ c_2 - \tfrac{1}{3} c_3 &= 0 \quad (2) \\ 0 &= 0 \quad (3) \end{aligned}}$$

leading: c_1, c_2

free: c_3

We get

$$\boxed{\begin{aligned} c_1 &= -c_2 - c_3 \\ c_2 &= \tfrac{1}{3} c_3 \\ c_3 &= t \end{aligned}}$$

1
2
3

Solving:

$$\boxed{\begin{aligned} &3 \quad c_3 = t \\ &2 \quad c_2 = \tfrac{1}{3} c_3 = \tfrac{1}{3} t \\ &1 \quad c_1 = -c_2 - c_3 = -\tfrac{1}{3} t - t = \tfrac{-4}{3} t \end{aligned}}$$

Plugging back into

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

gives

$$\left(-\frac{4}{3} t\right) \vec{v}_1 + \left(\frac{1}{3} t\right) \vec{v}_2 + (t) \vec{v}_3 = \vec{0}$$

for every t.

Plug in $t=1$ gives:

$$\boxed{-\frac{4}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_2 + 1\cdot\vec{v}_3 = \vec{0}}$$

So, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

<u>Ex:</u> $V = P_2$ ← polynomials of degree up to 2

$F = \mathbb{R}$

Let $\vec{v}_1 = 1$, $\vec{v}_2 = 1+x$, $\vec{v}_3 = 1+x+x^2$

Are these vectors lin. dep. or lin. ind. ?

We need to solve

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

which becomes

$$c_1(1) + c_2(1+x) + c_3(1+x+x^2) = \underbrace{0+0x+0x^2}_{\vec{0}}$$

which becomes

$$c_1 + c_2 + c_2 x + c_3 + c_3 x + c_3 x^2 = 0+0x+0x^2$$

which gives

$$(c_1+c_2+c_3) + (c_2+c_3)x + c_3 x^2 = 0+0x+0x^2$$

which gives

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 0 \quad (1) \\ c_2 + c_3 &= 0 \quad (2) \\ c_3 &= 0 \quad (3) \end{aligned}}$$

reduced with no free variables

Solving gives:

(3) $c_3 = 0$

(2) $c_2 = -c_3 = -0 = 0$

(1) $c_1 = -c_2 - c_3 = -0 - 0 = 0$

Thus the only solution to

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

is $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

So, $\vec{v}_1 = 1$, $\vec{v}_2 = 1 + x$, $\vec{v}_3 = 1 + x + x^2$

are linearly independent.