Math 2550-03

3/21/24

Topic 7 - Bases

This topic is about creating coordinate systems for vector spaces.

<u>Def:</u> Let V be a vector space over a field F. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be vectors in V.

(1) We say that a vector $\vec{v}$ is in the <u>span</u> of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if we can write

$$\vec{v} = \underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n}_{\text{this is called a linear combination of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n}$$

where $c_1, c_2, \ldots, c_n$ are scalars from F.

(2) The <u>span</u> of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is the set

$\text{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\})$ name of the set

$$= \underbrace{\left\{ c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \;\middle|\; \begin{array}{l} c_1, c_2, \ldots, c_n \\ \text{are in } F \end{array} \right\}}_{\text{set of all linear combos of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n}$$

(3) If $W = \text{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\})$ then we say that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

span W.

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

Let $\vec{v}_1 = \langle 1, 0 \rangle$.

Q: Is $\vec{w} = \langle 7, 0 \rangle$ in the span of $\vec{v}_1$?

That is, can we write

$$\underbrace{\langle 7, 0 \rangle = c_1 \langle 1, 0 \rangle}_{\vec{w} = c_1 \vec{v}_1}$$

Yes, $\langle 7, 0 \rangle = 7 \langle 1, 0 \rangle$

That is, $\vec{w} = 7\vec{v}_1$.

So, $\vec{w}$ is in the span of $\vec{v}_1$.

<u>Q:</u> Is $\vec{z} = \langle 6, -4 \rangle$ in the span of $\vec{v}_1 = \langle 1, 0 \rangle$?

That is, can we write

$$\langle 6, -4 \rangle = c_1 \langle 1, 0 \rangle \quad \longleftarrow \vec{z} = c_1 \vec{v}_1$$

No, because you would need

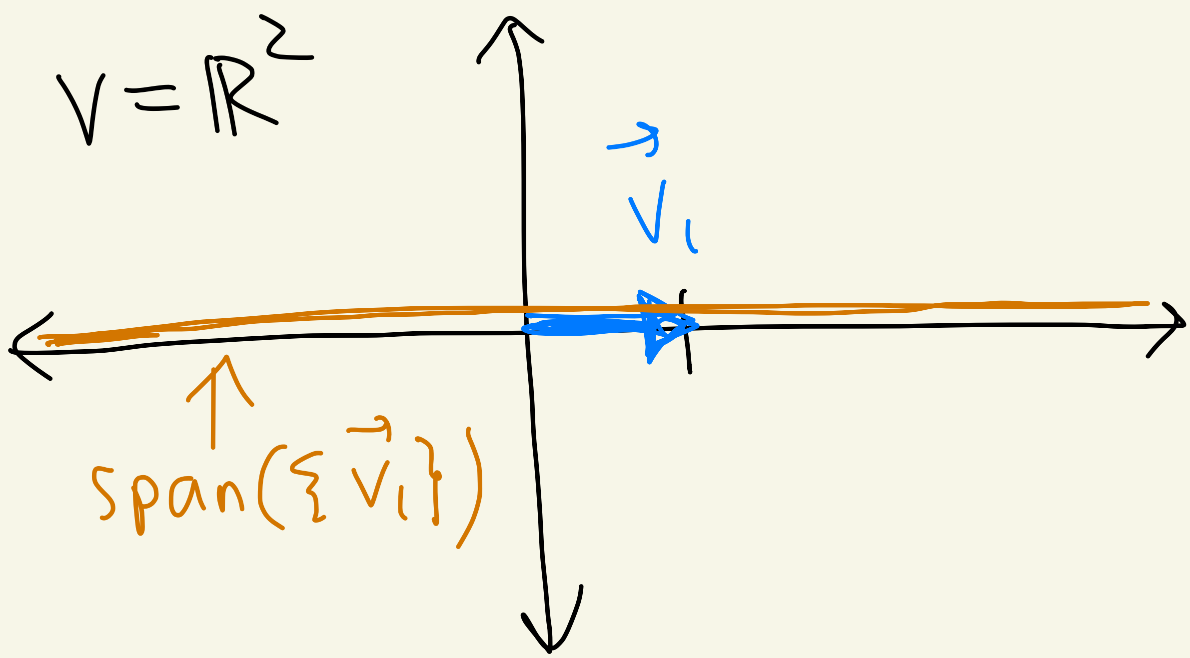
$$\langle 6, -4 \rangle = \langle c_1, 0 \rangle$$

no way since $-4 \neq 0$!

<u>Q:</u> What is the span of $\vec{v}_1 = \langle 1, 0 \rangle$?

$$\text{span}(\{\vec{v}_1\}) = \{c_1\vec{v}_1 \mid c_1 \in \mathbb{R}\}$$
$$= \{c_1\langle 1, 0\rangle \mid c_1 \in \mathbb{R}\}$$
$$= \{\langle c_1, 0\rangle \mid c_1 \in \mathbb{R}\}$$

So, the span of $\vec{v}_1$ consists of all vectors of the form $\langle c_1, 0\rangle$.



__Ex:__ Let $V = \mathbb{R}^2$, $F = \mathbb{R}$

Let $\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$.

__Q:__ Is $\vec{w} = \langle 2, -1 \rangle$ in the span of $\vec{v}_1, \vec{v}_2$?

Yes, because

$$\underbrace{\langle 2, -1 \rangle}_{\vec{w}} = \underbrace{2 \cdot \langle 1, 0 \rangle + (-1) \cdot \langle 0, 1 \rangle}_{c_1 \vec{v}_1 + c_2 \vec{v}_2}$$

$$\vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

$$\vec{w} = 2 \cdot \vec{v}_1 - 1 \cdot \vec{v}_2$$

__Q:__ What is the span of

$\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$?

$$\operatorname{Span}\left(\{\vec{v}_1, \vec{v}_2\}\right)$$

$$= \{ \underbrace{c_1\langle 1,0\rangle + c_2\langle 0,1\rangle}_{c_1\vec{v}_1 + c_2\vec{v}_2} \mid c_1, c_2 \in \mathbb{R} \}$$

$$= \{ \langle c_1, 0\rangle + \langle 0, c_2\rangle \mid c_1, c_2 \in \mathbb{R} \}$$

$$= \{ \langle c_1, c_2\rangle \mid c_1, c_2 \in \mathbb{R} \}$$

$$= \mathbb{R}^2$$

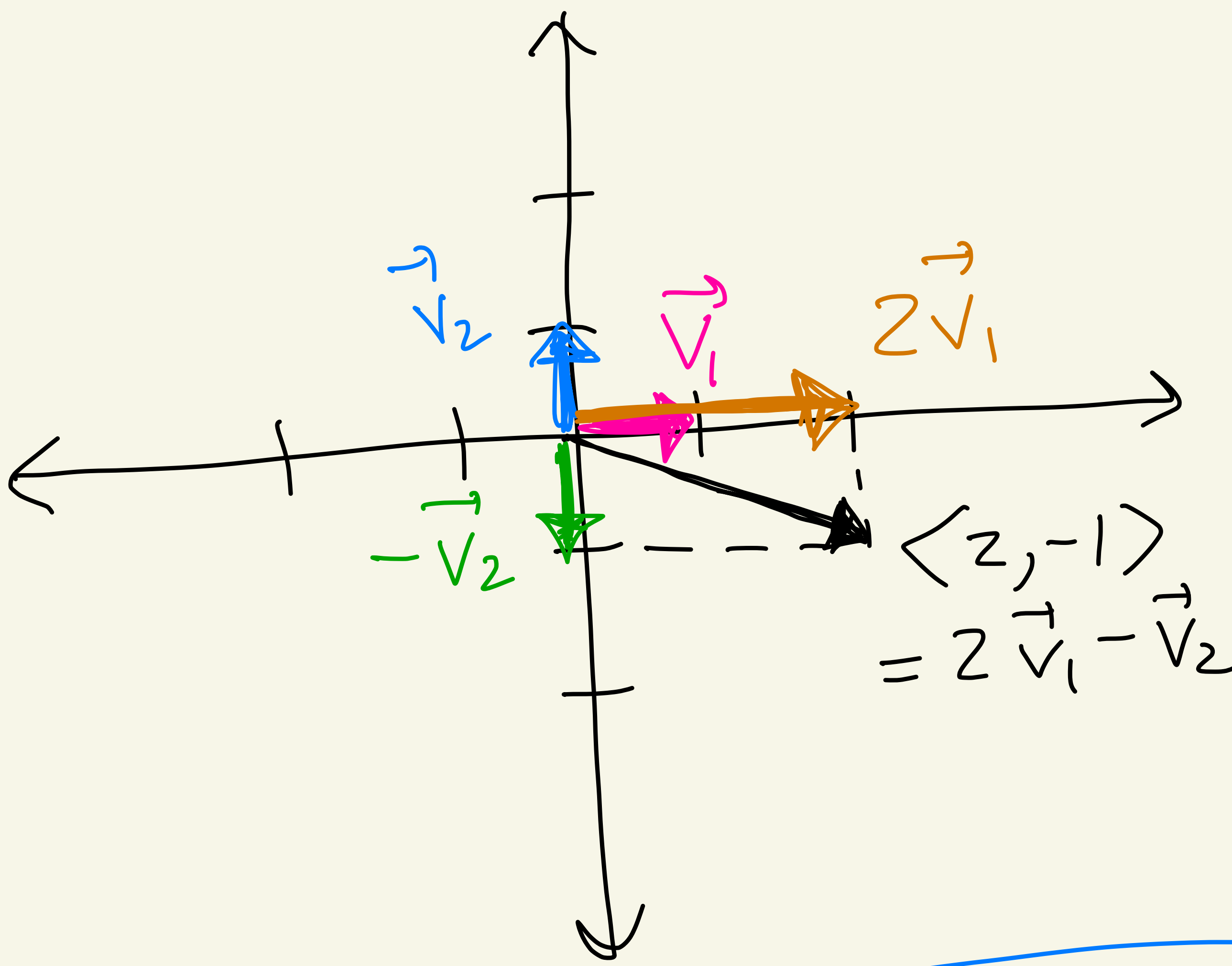
Another way to see that any vector $\langle a,b\rangle$ is in the span of $\vec{v}_1, \vec{v}_2$ is with the formula

$$\langle a,b\rangle = \underbrace{a\cdot\langle 1,0\rangle + b\cdot\langle 0,1\rangle}_{}$$

$$\langle a,b\rangle = a\cdot\vec{v}_1 + b\cdot\vec{v}_2$$

Thus, $\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$ span all of $\mathbb{R}^2$.

picture of $\langle 2, -1 \rangle = 2\vec{v}_1 - 1 \cdot \vec{v}_2$



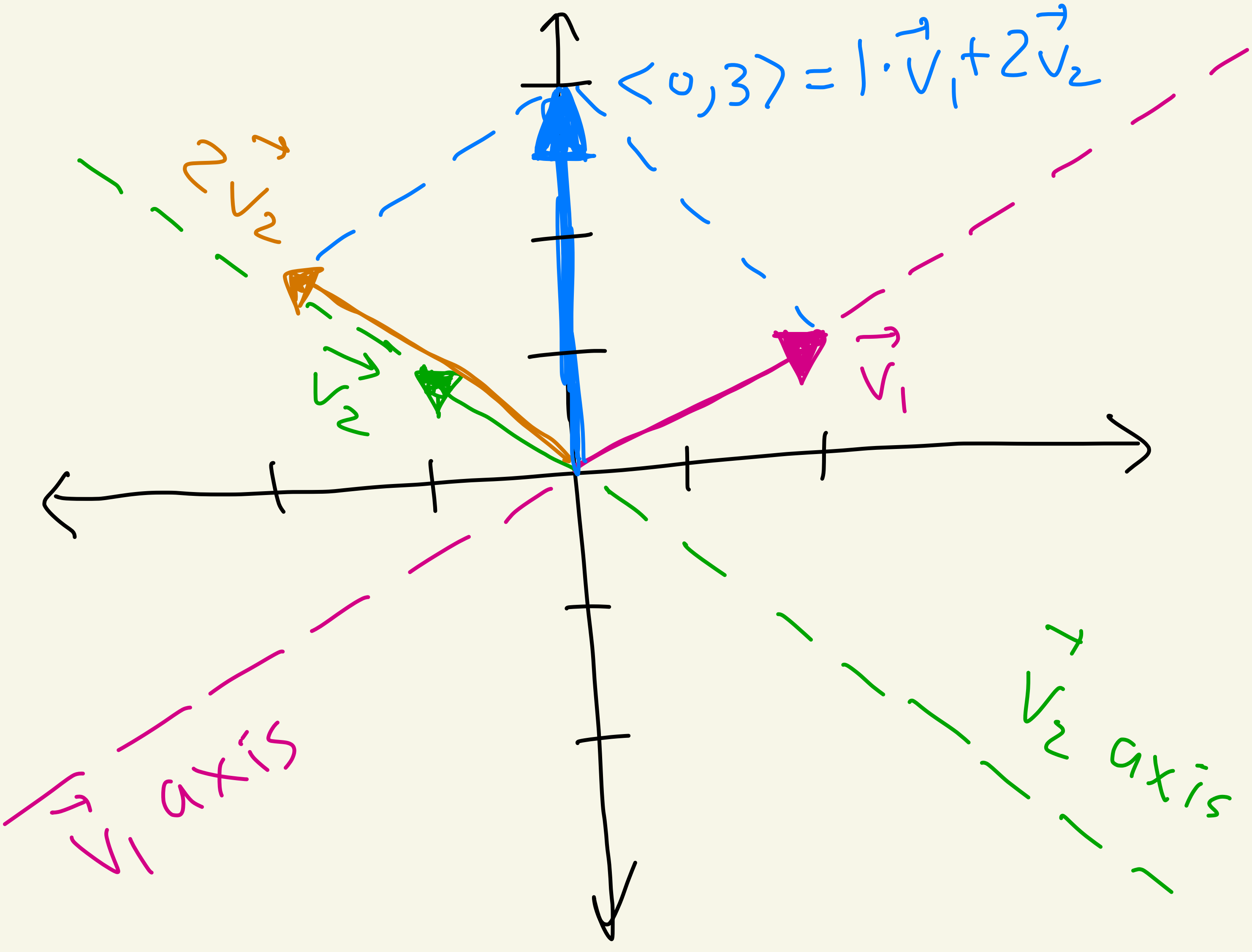
$\underline{\text{Ex:}}$ Let $V = \mathbb{R}^2$, $F = \mathbb{R}$

Let $\vec{v}_1 = \langle 2,1 \rangle$, $\vec{v}_2 = \langle -1,1 \rangle$

An example vector in the span of $\vec{v}_1, \vec{v}_2$ is

$$1 \cdot \vec{v}_1 + 2 \cdot \vec{v}_2 = \langle 2,1 \rangle + \langle -2,2 \rangle$$

$$= \langle 0,3 \rangle$$



Claim: Every vector in $\mathbb{R}^2$ is in the span of $\vec{v}_1 = \langle 2,1 \rangle$ and $\vec{v}_2 = \langle -1,1 \rangle$. That is, $\vec{v}_1, \vec{v}_2$ span all of $\mathbb{R}^2$

proof: Let $\langle a,b \rangle$ be some vector from $\mathbb{R}^2$. Let's show that can write

$$\langle a,b \rangle = \underbrace{c_1 \langle 2,1 \rangle + c_2 \langle -1,1 \rangle}_{c_1 \vec{v}_1 + c_2 \vec{v}_2}$$

This equation becomes

$$\langle a,b \rangle = \langle 2c_1, c_1 \rangle + \langle -c_2, c_2 \rangle$$

which gives

$$\langle a, b \rangle = \langle 2c_1 - c_2, c_1 + c_2 \rangle$$

This gives

$$\boxed{\begin{aligned} 2c_1 - c_2 &= a \\ c_1 + c_2 &= b \end{aligned}}$$

Let's solve:

$$\left(\begin{array}{cc|c} 2 & -1 & a \\ 1 & 1 & b \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 2 & -1 & a \end{array}\right)$$

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 0 & -3 & a-2b \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 0 & 1 & -\frac{1}{3}a + \frac{2}{3}b \end{array}\right)$$

This gives:

$$\boxed{\begin{aligned} c_1 + c_2 &= b \quad (1) \\ c_2 &= -\tfrac{1}{3}a + \tfrac{2}{3}b \quad (2) \end{aligned}}$$

(2) $c_2 = -\frac{1}{3}a + \frac{2}{3}b$

(1) $c_1 = b - c_2 = b - \left(-\frac{1}{3}a + \frac{2}{3}b\right)$

$= \frac{1}{3}a + \frac{1}{3}b$

Plugging back into

$$\langle a, b\rangle = c_1\langle 2, 1\rangle + c_2\langle -1, 1\rangle$$

gives

$$\boxed{\langle a, b\rangle = \left(\tfrac{1}{3}a + \tfrac{1}{3}b\right)\langle 2, 1\rangle + \left(-\tfrac{1}{3}a + \tfrac{2}{3}b\right)\langle -1, 1\rangle}$$

Thus, every vector $\langle a, b\rangle$ is in the span of $\vec{v}_1 = \langle 2, 1\rangle$, $\vec{v}_2 = \langle -1, 1\rangle$ $\boxed{\text{claim}}$

Ex: $\langle a, b\rangle = \langle 0, 2\rangle$

$$\langle 0,2\rangle = \left(\frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 2\right)\cdot\langle 2,1\rangle + \left(-\frac{1}{3}\cdot 0 + \frac{2}{3}\cdot 2\right)\cdot\langle -1,1\rangle$$

$$\langle 0,2\rangle = \frac{2}{3}\cdot\langle 2,1\rangle + \frac{4}{3}\cdot\langle -1,1\rangle$$

<u>Theorem:</u> Let V be a vector space over a field F. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be in V. Then, $\mathrm{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\})$ is a subspace of V.

$\boxed{\text{picture when } n=3}$