Math 2550-03
3/19/24

Ex: Let $F = \mathbb{R}$ ← scalars

and

vectors

$$V = P_2 = \{ a + bx + cx^2 \mid a, b, c \in \mathbb{R} \}$$

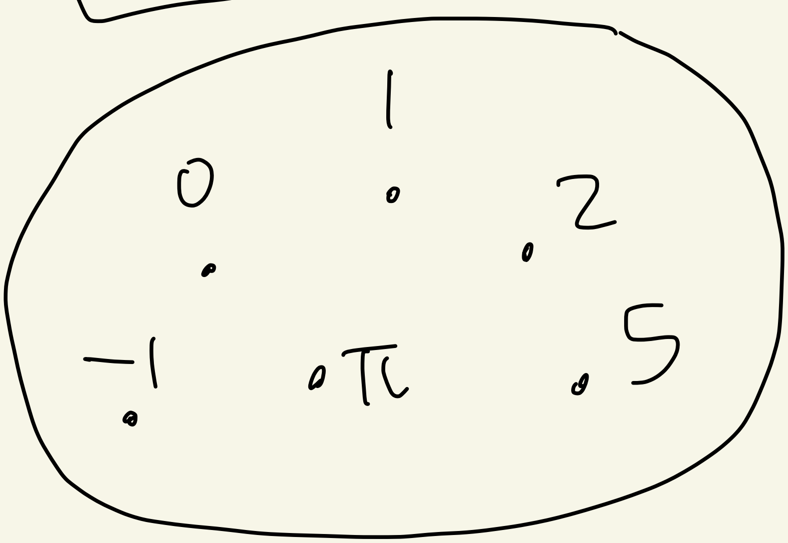
up to degree 2 polynomials

$x^2 = 0 + 0x + 1x^2$

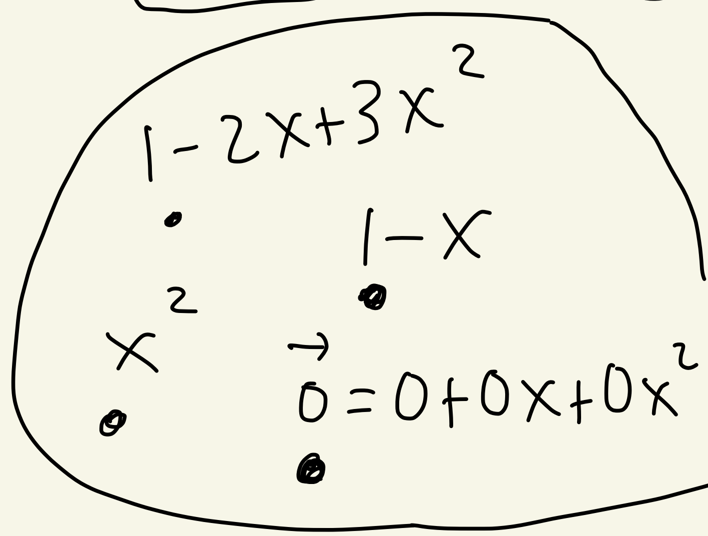
$$= \{ 1 - 2x + 3x^2, \; 7, \; x^2, \; 1 - x, \; \ldots \}$$

$7 = 7 + 0x + 0x^2$

$\boxed{\text{field}}$ $F = \mathbb{R}$



$\boxed{\text{vectors}}$ $V = P_2$



example of adding vectors:

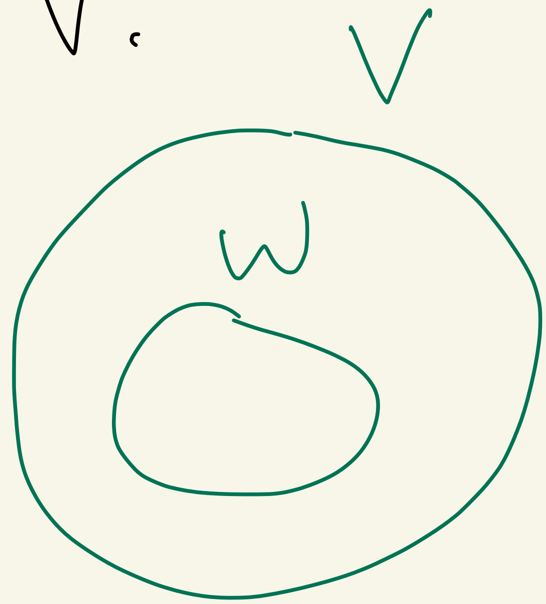
$$(1-2x+3x^2)+(1-x)=2-3x+3x^2$$

example of scaling a vector:

$$5\cdot(1-2x+3x^2)=5-10x+15x^2$$

Now we define what a subspace is, which ends up being a vector space inside of a vector space

Def: Let V be a vector space over a field F. Let W be a subset of V. We say that W is a **subspace** of V if

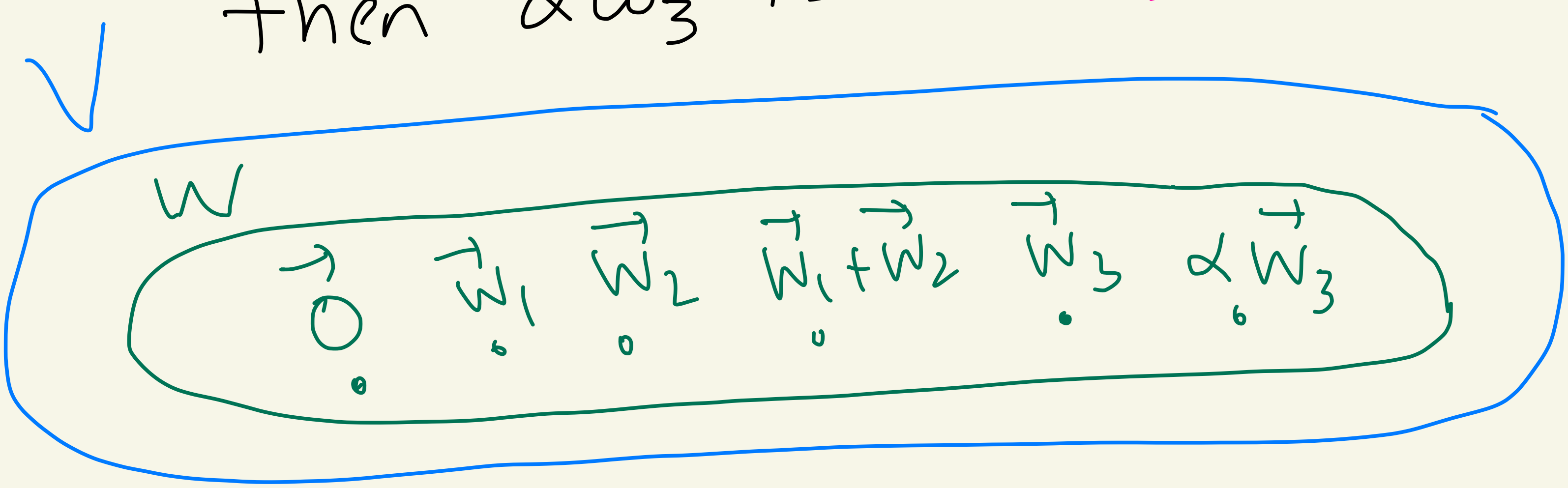


1. $\vec{0}$ is in W.
2. if $\vec{w}_1, \vec{w}_2$ are in W, then $\vec{w}_1 + \vec{w}_2$ is in W.

   W is closed under vector addition

3. if $\vec{w}_3$ is in W and α is a scalar in F, then $\alpha \vec{w}_3$ is in W.

   W is closed under scaling

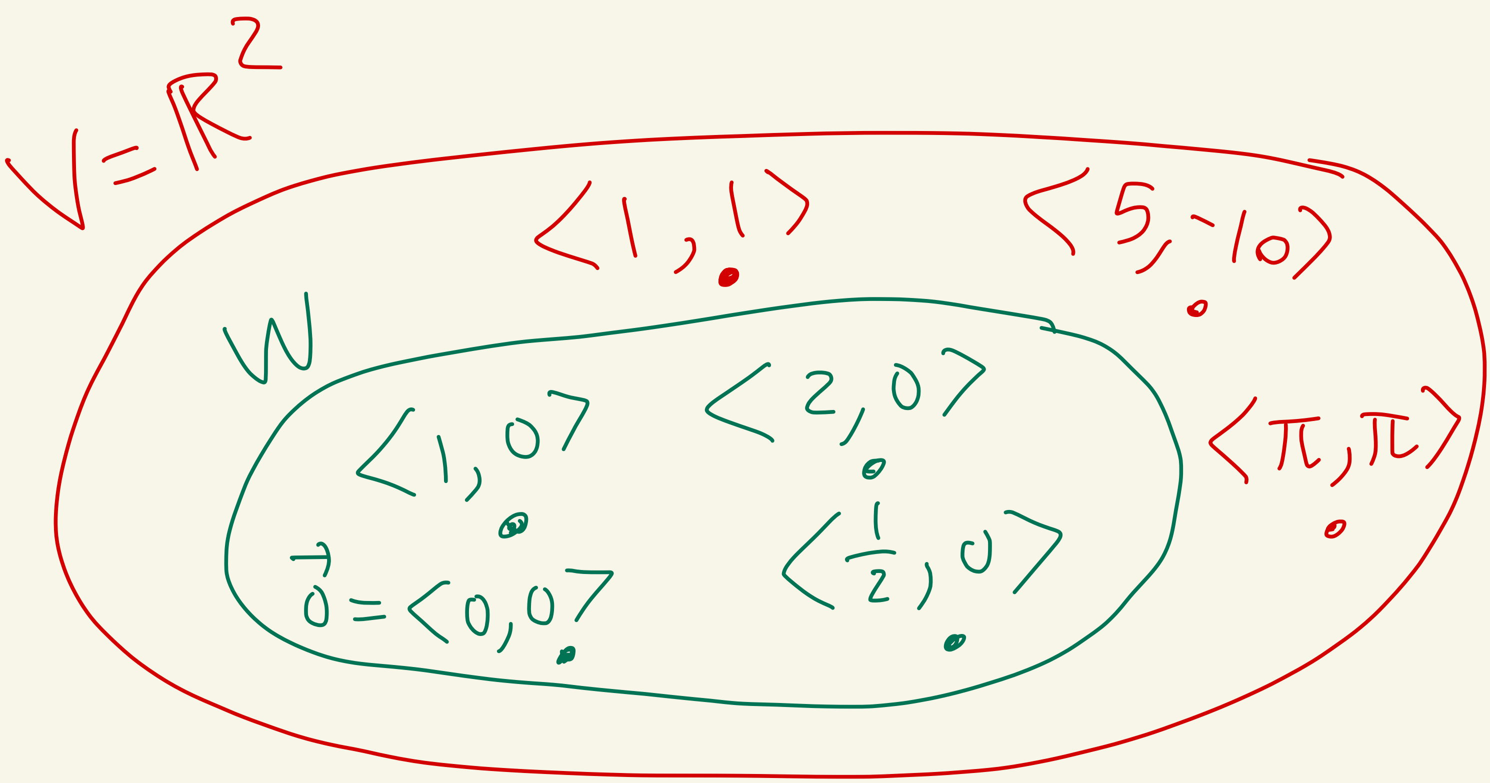


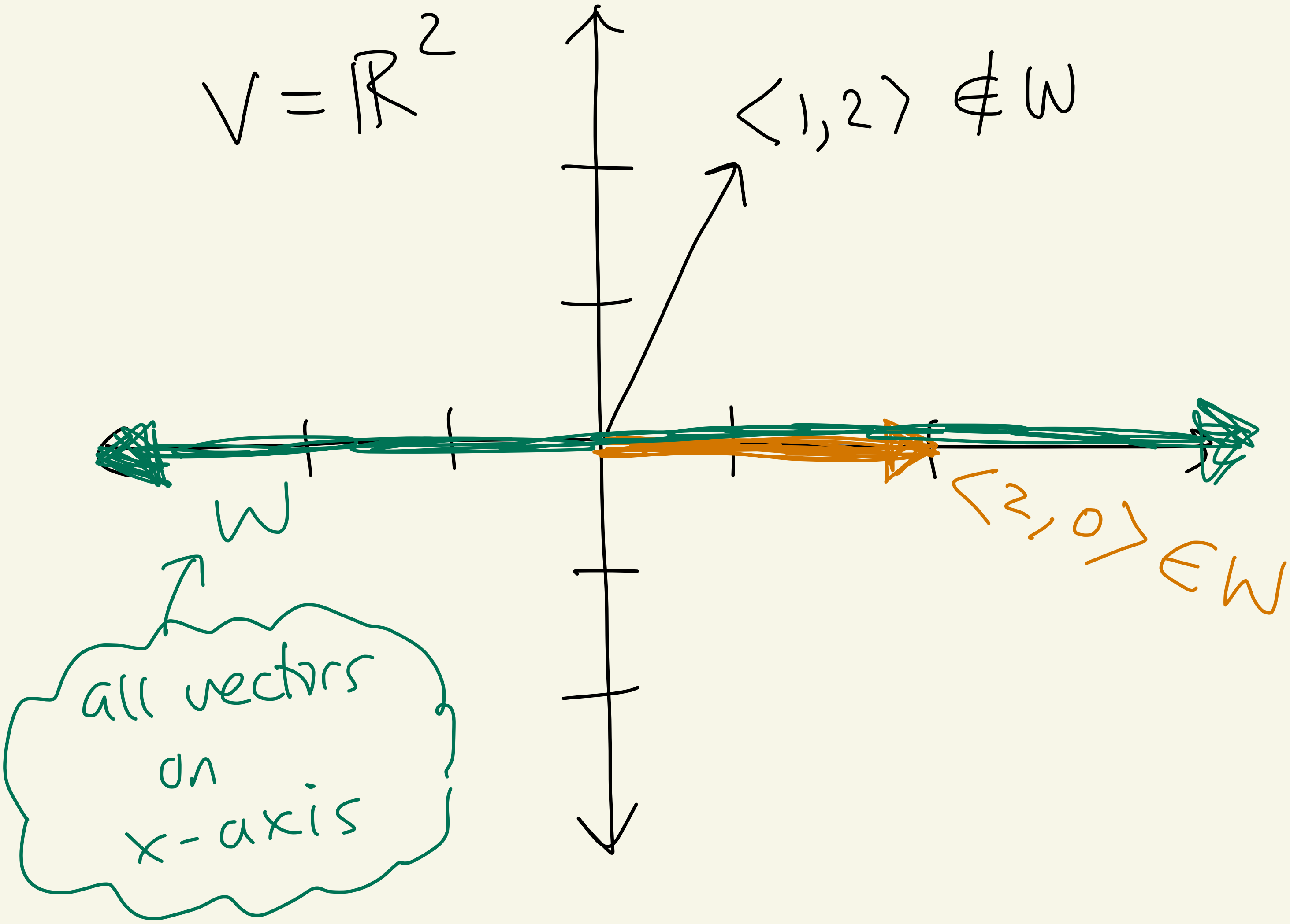
Ex: $\left.\begin{array}{l}\text{Let } F=\mathbb{R}\\ \text{and } V=\mathbb{R}^2.\end{array}\right]$ vector space

Let

$$W=\{\langle x,0\rangle \mid x\in\mathbb{R}\}$$

$$=\{\langle 1,0\rangle, \langle 2,0\rangle, \langle -3,0\rangle, \langle \tfrac{1}{2},0\rangle, \langle \pi,0\rangle, \ldots\}$$





$\boxed{W \text{ is a subspace.}}$

proof:

(1) Set $x=0$ in $\langle x,0 \rangle$ gives $\langle x,0 \rangle = \langle 0,0 \rangle = \vec{0}$

So, $\vec{0}$ is in W.

(2) Let $\vec{w}_1, \vec{w}_2$ be in W.
Then, $\vec{w}_1 = \langle x_1, 0 \rangle$ and
$\vec{w}_2 = \langle x_2, 0 \rangle$ where $x_1, x_2 \in \mathbb{R}$.

Then, $w_1 + w_2 = \langle x_1 + x_2, 0 \rangle$
is of the form $\langle x, 0 \rangle$
and so $\vec{w}_1 + \vec{w}_2$ is in W.

(3) Let $\vec{w}_3$ be in W
and α in $F = \mathbb{R}$.
Then, $\vec{w}_3 = \langle x_3, 0 \rangle$ where $x_3 \in \mathbb{R}$.
Then, $\alpha \vec{w}_3 = \alpha \langle x_3, 0 \rangle$

$$= \langle \alpha x_3, \alpha 0 \rangle$$

$$= \langle \alpha x_3, 0 \rangle$$

So, $\alpha \vec{w}_3$ is of the form $\langle x, 0 \rangle$ and thus $\alpha \vec{w}_3$ is in W.

By ①, ②, ③, W is a subspace. $\blacksquare$

We just showed W is itself a vector space inside of the bigger vector space $V = \mathbb{R}^2$. That's what subspace ends up meaning

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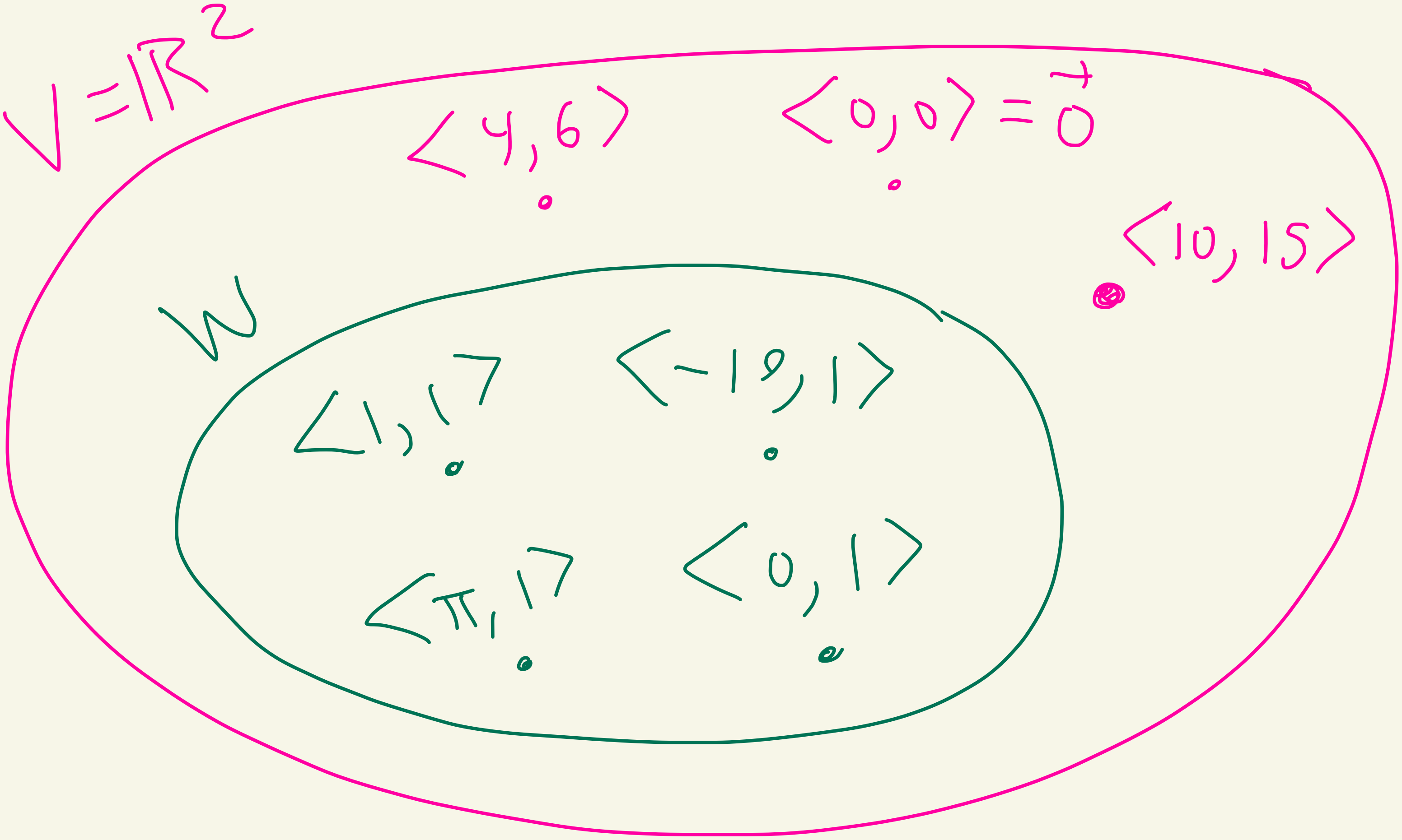
Ex: Let $F = \mathbb{R}$ and $V = \mathbb{R}^2$.] vector space.

Let

$$W = \{\langle x, 1\rangle \mid x \in \mathbb{R}\}$$
$$= \{\langle 1, 1\rangle, \langle -10, 1\rangle, \langle \pi, 1\rangle, \langle 0, 1\rangle, \ldots\}$$



To show that W is not a subspace, you give an example to show that either ①, ②, or ③ fails.

For example, $\vec{0} = \langle 0,0 \rangle$ is not of the form $\langle x, 1 \rangle$ and hence $\vec{0} \notin W$. So, ① fails and W is not a subspace.

Or you could say that ② fails, for example $\langle 2,1 \rangle$ and $\langle 3,1 \rangle$ are in W, but

$$\langle 2,1 \rangle + \langle 3,1 \rangle = \langle 5,2 \rangle$$

and $\langle 5,2 \rangle$ is not in W. So W is not a subspace.

$\boxed{\text{Ex:}}$ Let $F = \mathbb{R}$

and $V = M_{2,2}$ ← set of all 2×2 matrices

Let

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} d = a+b \\ a,b,c,d \in \mathbb{R} \end{array} \right\}$$

$$= \left\{ \underbrace{\begin{pmatrix} 5 & 3 \\ 1 & 8 \end{pmatrix}}_{\substack{\text{in } W \\ \text{since } 8=5+3}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\substack{\text{in } W \\ \text{since} \\ 0=0+0}}, \underbrace{\begin{pmatrix} -1 & -2 \\ 5 & -3 \end{pmatrix}}_{\substack{\text{in } W \\ \text{since} \\ -3=-1+(-2)}}, \ldots \right\}$$

Calculations:

$$\underbrace{\begin{pmatrix} 5 & 3 \\ 1 & 8 \end{pmatrix}}_{\text{in } W} + \underbrace{\begin{pmatrix} -1 & -2 \\ 5 & -3 \end{pmatrix}}_{\text{in } W} = \underbrace{\begin{pmatrix} 4 & 1 \\ 6 & 5 \end{pmatrix}}_{\text{in } W}$$

$$2 \cdot \underbrace{\begin{pmatrix} 5 & 3 \\ 1 & 8 \end{pmatrix}}_{\text{in } W} = \underbrace{\begin{pmatrix} 10 & 6 \\ 2 & 16 \end{pmatrix}}_{\text{in } W}$$

vs trying to see if we think W is a subspace or not

Let's prove that W is a subspace

① Setting $a=0, b=0, c=0, d=0$

gives $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \vec{0}$

and $d = a+b$.

So, $\vec{0}$ is in W.

② Let $\vec{w}_1, \vec{w}_2 \in W$.

Then, $\vec{w}_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ where $\boxed{d_1 = a_1 + b_1}$

and $\vec{w}_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ where $\boxed{d_2 = a_2 + b_2}$

Then,

$$\vec{w}_1 + \vec{w}_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$ and

$$d_1 + d_2 = (a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2)$$

So, $\vec{w}_1 + \vec{w}_2$ is in W.

③ Let $\vec{w}_3$ be in W and α be a scalar in $F = \mathbb{R}$.

Then, $\vec{w}_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$

where $\boxed{d_3 = a_3 + b_3}$.

Thus,

$$\alpha \vec{w}_3 = \begin{pmatrix} \alpha a_3 & \alpha b_3 \\ \alpha c_3 & \alpha d_3 \end{pmatrix}$$

and $\alpha d_3 = \alpha(a_3 + b_3) = \alpha a_3 + \alpha b_3$

Thus, $\alpha \vec{w}_3$ is in W.

By ①, ②, ③, W is a subspace of $V = M_{2,2}$