Math 2550-03
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$\pi \approx \underline{3.14}159\ldots$

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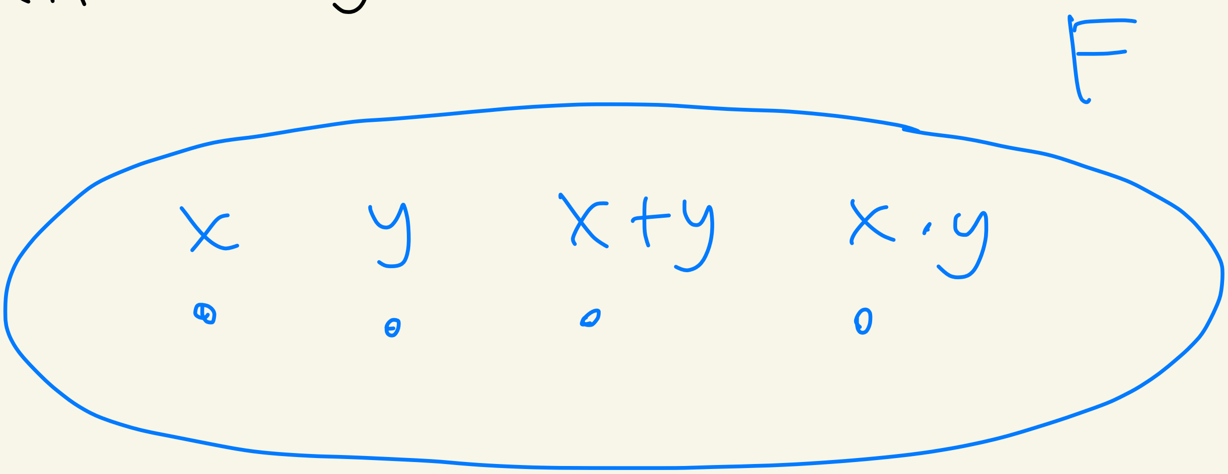
Topic 6 – Vector Spaces

field

We are going to generalize
What a scalar/number is
and what a vector is

Vector space

Def: A field consists of a set F of "scalars/numbers" and two operations $+$ and $\cdot$ such that if x and y are in F then $x+y$ and $x \cdot y$ are in F.



The following properties must also hold:

F1 If a, b, c are in F, then:

$$a+b=b+a$$

$$a \cdot b = b \cdot a$$

$$a+(b+c)=(a+b)+c$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$(b + c) \cdot a = b \cdot a + c \cdot a$$

F2 There exist unique elements 0 and 1 in F where

$$x + 0 = 0 + x = x$$

and $x \cdot 1 = 1 \cdot x = x$

for all x in F.

F3 Let x be in F.

There exists a unique element $-x$ in F where

$$x + (-x) = (-x) + x = 0.$$

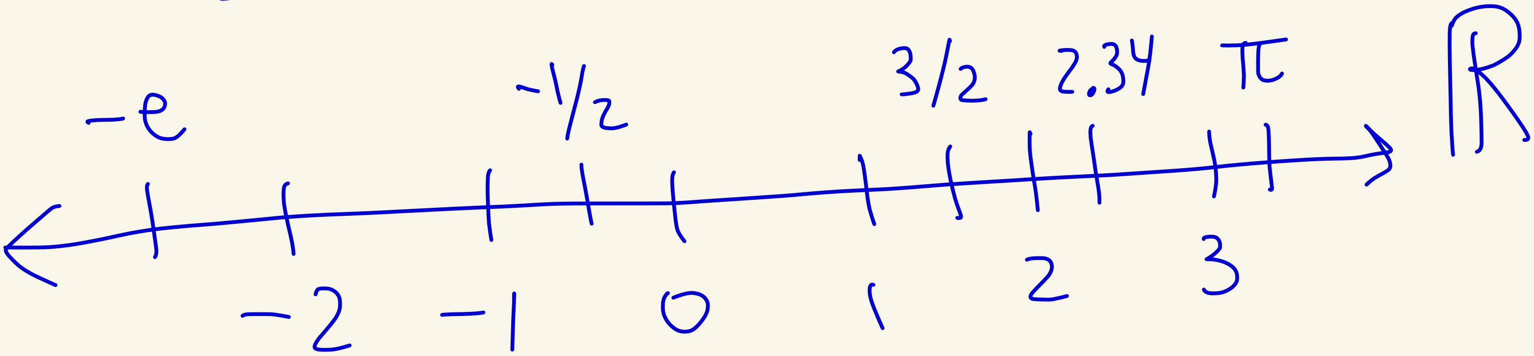
If $x \neq 0$, then there exists

a unique element x^{-1} in F
where $x \cdot x^{-1} = x^{-1} \cdot x = 1$

END OF DEF

<u>Ex:</u> $F = \mathbb{R}$ ← set of real numbers

$\mathbb{R}$ is a field using the usual adding and multiplying.

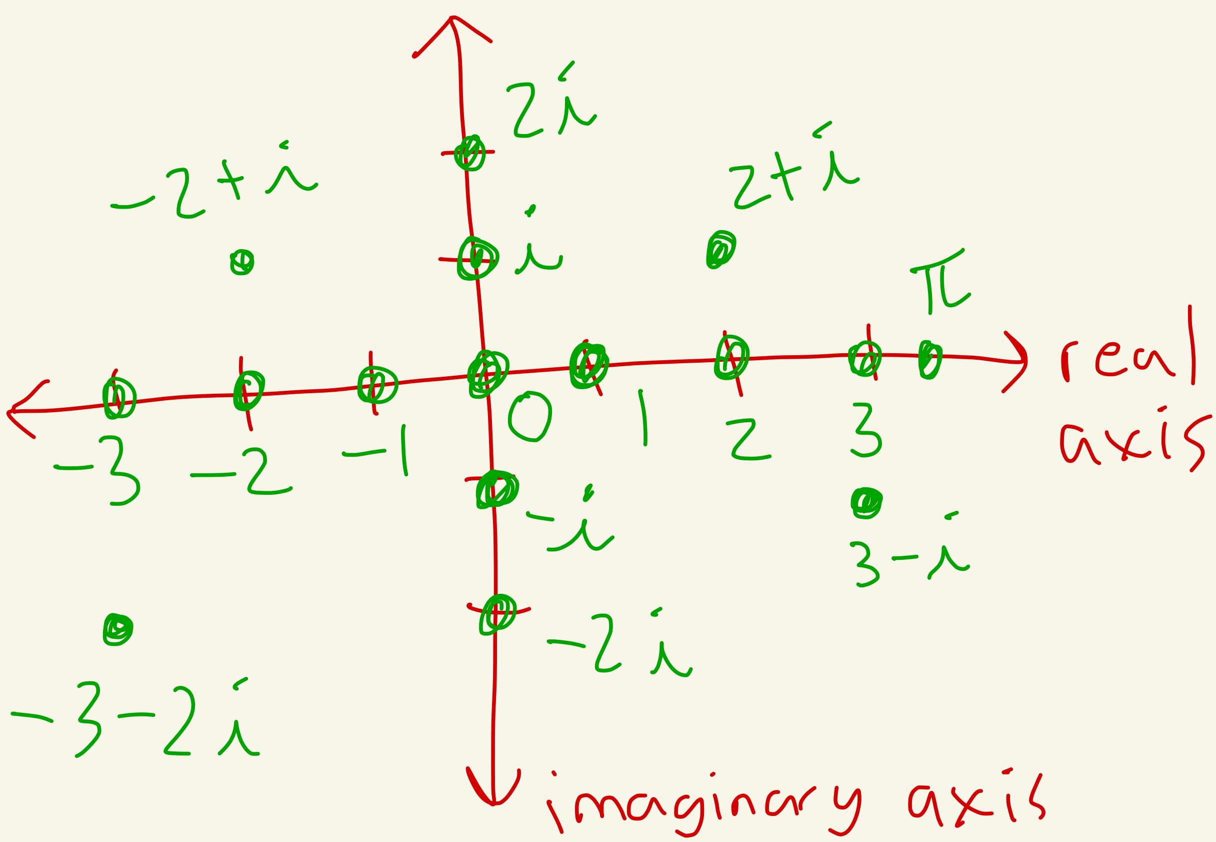


In this class $\mathbb{R}$ will be the only field we use, but I will state definitions and theorems for general fields.

<u>Ex:</u> The set of complex numbers $\mathbb{C}$ is also a field.

A complex number is a number of the form $a+bi$ where a & b are real numbers

And $i^2=-1$ or $i=\sqrt{-1}$



Ex: There are finite fields that are built with prime numbers and modular arithmetic

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Now let's generalize vectors.

Def: A **vector space V over a field F** consists of a set of "vectors" V and a field F with two operations, "vector addition" $+$ and "vector scaling" $\cdot$ such that if $\vec{v}, \vec{w}, \vec{z}$ are vectors from V and α, β are scalars from F, then the following must hold:

① $\vec{v} + \vec{w}$ is in V

② $\alpha \cdot \vec{v}$ is in V

adding vectors & scaling a vector gives a vector

③ $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

④ $\vec{v} + (\vec{w} + \vec{z}) = (\vec{v} + \vec{w}) + \vec{z}$

(5) there exists a unique vector $\vec{0}$ in V, called the zero vector, where $\vec{0}+\vec{y}=\vec{y}+\vec{0}=\vec{y}$ for any vector $\vec{y}$ from V.

(6) there exists a unique vector $-\vec{v}$ in V where

$$\vec{v}+(-\vec{v})=(-\vec{v})+\vec{v}=\vec{0}$$

(7) $1\cdot\vec{v}=\vec{v}$

(8) $(\alpha\beta)\cdot\vec{v}=\alpha\cdot(\beta\cdot\vec{v})$

(9) $\alpha\cdot(\vec{v}+\vec{w})=\alpha\cdot\vec{v}+\alpha\cdot\vec{w}$

(10) $(\alpha+\beta)\cdot\vec{v}=\alpha\cdot\vec{v}+\beta\cdot\vec{v}$

END OF DEF

<u>Ex:</u> Let $V = \mathbb{R}^n$ and $F = \mathbb{R}$.

Then, $V = \mathbb{R}^n$ is a vector space over $F = \mathbb{R}$ using the usual vector adding and scaling.

$\boxed{n=2 \text{ example}}$

$\boxed{\text{Field}}$

$F = \mathbb{R}$

$0 \quad 1 \quad 2 \quad -\pi \quad 1/2$

$\boxed{\text{vectors}}$

$V = \mathbb{R}^2$

$\langle 0,0 \rangle = \vec{0} \quad \langle 5, 1/2 \rangle \quad \langle 1,0 \rangle \quad \langle 3,7 \rangle$

adding: $\langle a,b \rangle + \langle c,d \rangle = \langle a+c, b+d \rangle$

scaling: $\alpha \langle a,b \rangle = \langle \alpha a, \alpha b \rangle$

Ex: Let $V = M_{2,2}$ be the set of 2x2 matrices with real number entries.

So,

$$V = M_{2,2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & \pi \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0.7 & 1 \end{pmatrix}, \ldots \right\}$$

$$\boxed{\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}$$

infinitely many more

Let $F = \mathbb{R}$.

Then, $V = M_{2,2}$ is a vector space over $F = \mathbb{R}$

using the usual matrix adding and scaling.

field

$F = \mathbb{R}$

$1 \quad \frac{1}{2} \quad 0 \quad \pi \quad 10$

vectors

$V = M_{2,2}$

$\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 \\ 2 & 5 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 4 \end{pmatrix}$

adding: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$

scaling: $\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$

Ex: Let $n \geq 0$ be an integer.
[So, n can be $0, 1, 2, 3, 4, \ldots$]

Let $V = P_n$ be the set of all polynomials of degree $\leq n$.

So,

"vectors"

$$V = P_n = \{ a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \mid a_0, a_1, \ldots, a_n \text{ are real numbers} \}$$

Let $F = \mathbb{R}$.

Add and scale just like usual.

Adding vectors:

$$(a_0 + a_1 x + \ldots + a_n x^n) + (b_0 + b_1 x + \ldots + b_n x^n)$$

$$= (a_0 + b_0) + (a_1 + b_1) x + \ldots + (a_n + b_n) x^n$$

Scaling:

$$\alpha(a_0 + a_1 x + \dots + a_n x^n)$$
$$= (\alpha a_0) + (\alpha a_1)x + \dots + (\alpha a_n)x^n$$

Equality:

$$a_0 + a_1 x + \dots + a_n x^n = b_0 + b_1 x + \dots + b_n x^n$$

if and only if

$$a_0 = b_0,\ a_1 = b_1, \dots, a_n = b_n$$

Zero vector:

$$\vec{0} = 0 + 0x + 0x^2 + \dots + 0x^n$$

Fact: $V = P_n$ is a vector space over $F = \mathbb{R}$