Math 2550-03
2/29/24

(topic 4 continued...)

Last time we showed how to write a linear system in the form $A\vec{x} = \vec{b}$.

If A^{-1} exists then you can do this:

$$A\vec{x} = \vec{b}$$

$$\underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = \vec{x}$$

$$\vec{x} = A^{-1}\vec{b}$$

So, if A^{-1} exists then $\vec{x} = A^{-1}\vec{b}$ is the only solution to our system.

$\boxed{\text{Ex:}}$ Find all the solutions to

$$\boxed{\begin{aligned} 3x \phantom{{}+y} + 3z &= 9 \\ x + y + 2z &= -4 \\ -2x + 3y \phantom{{}+2z} &= 5 \end{aligned}} \quad (*)$$

Write $(*)$ as $A\vec{x} = \vec{b}$.

$$A = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

Check that $A\vec{x} = \vec{b}$ encodes $(*)$

$$\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix} \leftarrow \boxed{A\vec{x} = \vec{b}}$$

$$\begin{pmatrix} 3x + 3z \\ x + y + 2z \\ -2x + 3y \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix} \leftarrow \boxed{\text{same as } (*)}$$

Thus $(*)$ is equivalent to

$$\underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{A\vec{x}} = \underbrace{\begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{\vec{b}}$$

Previously, we showed that

$$A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$$

Multiply by A^{-1} on the left gives

$$\underbrace{\overbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}^{I_3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{A^{-1}A\vec{x}} = \underbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}\begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{A^{-1}\vec{b}}$$

So we get

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix} \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

This becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (2)(9)+(-3)(-4)+(1)(5) \\ (4/3)(9)+(-2)(-4)+(1)(5) \\ (-5/3)(9)+(3)(-4)+(-1)(5) \end{pmatrix}$$

This will give

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 35 \\ 25 \\ -32 \end{pmatrix}$$

So the only answer to (*) is

$x = 35,\ y = 25,\ z = -32.$

Topic 5 – Determinants

The determinant will allow us to detect when a square matrix has an inverse.

<u>Def:</u> Let A be an $n \times n$ matrix. The matrix A_{ij} is defined to be the $(n-1) \times (n-1)$ matrix obtained by removing row i and column j from A

<u>Ex:</u>

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} \leftarrow \boxed{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = A}$$

$$A_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} \leftarrow \boxed{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = A}$$

Factorial is a recursive function

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot ((n-1)!) & \text{if } n \geq 1 \end{cases}$$

$$2! = 2 \cdot 1! = 2 \cdot 1 \cdot 0! = 2 \cdot 1 \cdot 1 = 2$$

Determinant is also recursive!

Def: Let A be an $n \times n$ matrix. Let a_{ij} be the number in A located at row i, column j. Define the determinant of A, denoted by $\det(A)$, as follows:

① If $n=1$ and $A = (a_{11})$, then $\det(A) = a_{11}$.

② If $n=2$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

then $$\underbrace{\det(A) = a_{11}a_{22} - a_{21}a_{12}}_{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}$$

③ If $n \geq 3$, then pick a column j to "expand on" and define

$$\det(A) = \underbrace{\sum_{i=1}^{n}}_{\text{sum over rows } i \text{ with column } j \text{ fixed}} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

This called expanding on column j.

Note: In step 3, you can instead expand on a row i and calculate

$$\det(A) = \underbrace{\sum_{j=1}^{n}}_{\text{sums over the columns } j \text{ row } i \text{ is fixed}} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

Note: It doesn't matter what row or column you pick in step 3, you'll always get the same answer at the end.

Note: Another notation for det is using bars like absolute value. Like this:

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

Ex:

$$\det(11) = 11$$

$$|11| = 11$$

<u>Ex:</u>

$$\det\begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix} = \underbrace{(1)(-6)-(3)(-2)}_{\begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}}$$

$$= 0$$

Ex:

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$j = 2$

Expand on column $j=2$

$$\det(A) = \underbrace{\sum_{i=1}^{3}}_{\substack{\text{sum over rows } i \\ \text{keep column } j=2}} (-1)^{i+2} \cdot a_{i2} \cdot \det(A_{i2})$$

$$= (-1)^{1+2} \cdot a_{12} \cdot \det(A_{12})$$ ← $i=1$ term

$$+ (-1)^{2+2} \cdot a_{22} \cdot \det(A_{22})$$ ← $i=2$ term

$$+ (-1)^{3+2} \cdot a_{32} \cdot \det(A_{32})$$ ← $i=3$ term

$$
= (-1)(1)\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}
$$

$$
+ (1)(-4)\begin{vmatrix} 3 & 0 \\ 5 & -2 \end{vmatrix} \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}
$$

$$
+ (-1)(4)\begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix} \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}
$$

$$
= (-1)\cdot\left[(-2)(-2) - (3)(5)\right]
$$

$$
+ (-4)\cdot\left[(3)(-2) + (0)(5)\right]
$$

$$
+ (-4)\cdot\left[(3)(3) + (0)(-2)\right]
$$

$$
= -1
$$

PICTURE WAY TO DETERMINE $(-1)^{i+j}$

$$\underbrace{\begin{pmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{pmatrix}}_{(-1)^{i+j}} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

i is row

j is column

we did this column $j = 2$