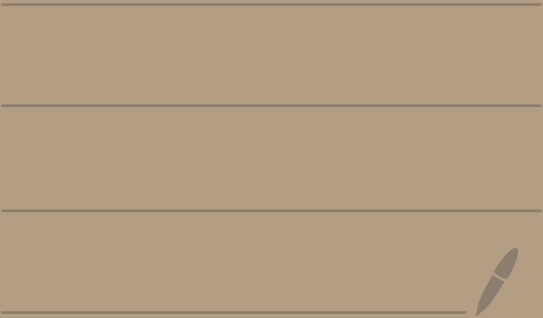


Math 2550-03

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## (topic 4 continued...)

Last time we showed how to write a linear system in the form  $A\vec{x} = \vec{b}$ .

If  $A^{-1}$  exists then you can do this:

$$\begin{array}{l} \vec{I} \vec{x} \\ = \vec{x} \end{array} \quad \begin{array}{l} \nearrow \\ \searrow \end{array} \quad \begin{array}{l} A\vec{x} = \vec{b} \\ \underline{A^{-1}A}\vec{x} = A^{-1}\vec{b} \\ \underline{I} \\ \vec{x} = A^{-1}\vec{b} \end{array}$$

So, if  $A^{-1}$  exists then

$\vec{x} = A^{-1}\vec{b}$  is the only solution to our system.

**Ex:** Find all the solutions to

$$\begin{aligned} 3x + 3z &= 9 \\ x + y + 2z &= -4 \\ -2x + 3y &= 5 \end{aligned} \quad (*)$$

Write (\*) as  $A\vec{x} = \vec{b}$ .

$$A = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

Check that  $A\vec{x} = \vec{b}$  encodes (\*)

$$\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix} \quad \leftarrow \boxed{A\vec{x} = \vec{b}}$$

$$\begin{pmatrix} 3x + 3z \\ x + y + 2z \\ -2x + 3y \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix} \quad \leftarrow \boxed{\text{same as } (*)}$$

Thus (\*) is equivalent to

$$\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

$A \vec{x} = \vec{b}$

Previously, we showed that

$$A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$$

Multiply by  $A^{-1}$  on the left gives

$$\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix} \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

$A^{-1} A \vec{x} = A^{-1} \vec{b}$

So we get

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix} \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

This becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (2)(9) + (-3)(-4) + (1)(5) \\ (4/3)(9) + (-2)(-4) + (1)(5) \\ (-5/3)(9) + (3)(-4) + (-1)(5) \end{pmatrix}$$

This will give

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 35 \\ 25 \\ -32 \end{pmatrix}$$

So the only answer to (\*) is

$$x = 35, y = 25, z = -32.$$

## Topic 5 - Determinants

The determinant will allow us to detect when a square matrix has an inverse.

---

Def: Let  $A$  be an  $n \times n$  matrix.

The matrix  $A_{\bar{i}j}$  is defined to be the  $(n-1) \times (n-1)$  matrix obtained by removing row  $i$  and column  $j$  from  $A$ .

---

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}$$

$$\left[ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = A \right]$$

$$A_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$\left[ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = A \right]$$

Factorial is a recursive function

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1 \end{cases}$$

$$2! = 2 \cdot 1! = 2 \cdot 1 \cdot 0! = 2 \cdot 1 \cdot 1 = 2$$

Determinant is also recursive!

Def: Let  $A$  be an  $n \times n$  matrix.

Let  $a_{ij}$  be the number in  $A$  located at row  $i$ , column  $j$ .

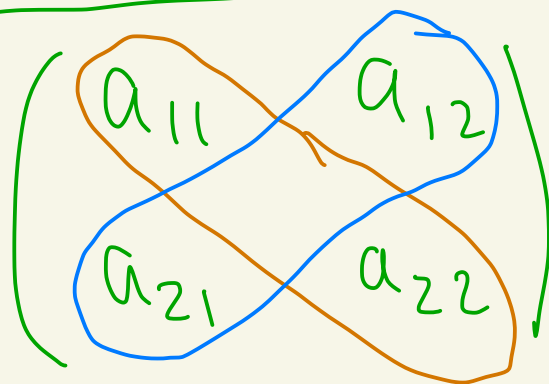
Define the determinant of  $A$ , denoted by  $\det(A)$ , as follows:

① If  $n=1$  and  $A = (a_{11})$ ,

then  $\det(A) = a_{11}$ .

② If  $n=2$  and  $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

then  $\det(A) = a_{11}a_{22} - a_{21}a_{12}$





③ If  $n \geq 3$ , then pick a column  $j$  to "expand on" and define

$$\det(A) = \sum_{\bar{i}=1}^n (-1)^{\bar{i}+\bar{j}} \cdot a_{\bar{i}\bar{j}} \cdot \det(A_{\bar{i}\bar{j}})$$

sum over rows  $\bar{i}$   
with column  $\bar{j}$  fixed

This called expanding on column  $\bar{j}$ .

---

Note: In step 3, you can instead expand on a row  $\bar{i}$  and calculate

$$\det(A) = \sum_{\bar{j}=1}^n (-1)^{\bar{i}+\bar{j}} \cdot a_{\bar{i}\bar{j}} \cdot \det(A_{\bar{i}\bar{j}})$$

sums over the columns  $\bar{j}$   
row  $\bar{i}$  is fixed

Note: It doesn't matter what row or column you pick in step 3, you'll always get the same answer at the end.

---

Note: Another notation for det is using bars like absolute value.

Like this:

$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

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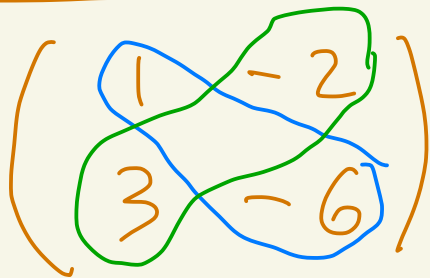
Ex:

$$\det(11) = 11$$

$$\begin{vmatrix} 11 \end{vmatrix} = 11$$

Ex:

$$\det \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix} = (1)(-6) - (3)(-2)$$



A diagram of a 2x2 matrix  $\begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}$  enclosed in large orange parentheses. A green loop connects the top-left element (1) to the bottom-right element (-6). A blue loop connects the top-right element (-2) to the bottom-left element (3). This illustrates the calculation of the determinant as the difference between the products of the two paths.

$$= 0$$

Ex:  $A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

↑  
 $\bar{j}=2$

Expand on column  $j=2$

$$\det(A) = \sum_{\bar{i}=1}^3 (-1)^{\bar{i}+2} \cdot a_{\bar{i}2} \cdot \det(A_{\bar{i}2})$$

sum over rows  $\bar{i}$   
keep column  $\bar{j}=2$

$$= (-1)^{1+2} \cdot a_{12} \cdot \det(A_{12})$$

←  $\bar{i}=1$   
term

$$+ (-1)^{2+2} \cdot a_{22} \cdot \det(A_{22})$$

←  $\bar{i}=2$   
term

$$+ (-1)^{3+2} \cdot a_{32} \cdot \det(A_{32})$$

←  $\bar{i}=3$   
term

$$= (-1)(1) \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} \quad \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$+ (1)(-4) \begin{vmatrix} 3 & 0 \\ 5 & -2 \end{vmatrix} \quad \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$+ (-1)(4) \begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix} \quad \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$= (-1) \cdot [(-2)(-2) - (3)(5)] \\ + (-4) \cdot [(3)(-2) + (0)(5)] \\ + (-4) \cdot [(3)(3) + (0)(-2)]$$

$$= -1$$

# PICTURE WAY TO DETERMINE $(-1)^{i+j}$

$$\begin{pmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$(-1)^{i+j}$

$i$  is row

$j$  is column

↑  
we did  
this  
column  
 $j=2$