Math 2550-03
2/27/24

__Fact:__ If A is $n \times n$ and has an inverse, then there exists only one inverse for A and we will call it A^{-1}.

__Ex:__ (Last time)

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \leftarrow \text{B from last time}$$

$$\left[AA^{-1} = A^{-1}A = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I_2} \leftarrow \boxed{\text{last time}} \right]$$

Procedure to find A^{-1} if it exists

Let A be an $n \times n$ matrix.

Start with the matrix

$$\left(A \mid I_n \right)$$

Then row reduce this matrix till either the left side has a row of zeros or I_n.

If you get a row of zeros on the left side, then A^{-1} does not exist.

If you get I_n on the left side, then the right side will have A^{-1} in it.

Ex: Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$.

Find A^{-1} if it exists.

$$\left(\begin{array}{cc|cc} \multicolumn{2}{c}{\overbrace{\qquad\quad}^{A}} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{I_2}} \\ 1 & 1 & 1 & 0 \\ \boxed{2} & 1 & 0 & 1 \end{array}\right) \xrightarrow{-2R_1+R_2 \to R_2}$$

make this 0

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & \boxed{-1} & -2 & 1 \end{array}\right)$$

make this 1

$$\xrightarrow{-R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & \boxed{1} & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

make 0

$$\xrightarrow{-R_2+R_1 \to R_1} \left(\begin{array}{cc|cc} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} & \underbrace{\begin{matrix} -1 & 1 \\ 2 & -1 \end{matrix}}_{A^{-1}} \end{array}\right)$$

So if $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ then

$$A^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

Ex: Find A^{-1} if it exists

when $A = \begin{pmatrix} 1 & 5 \\ -2 & -10 \end{pmatrix}$

$$\left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & 5 \\ \boxed{-2} & -10 \end{matrix}}^{A} & \overbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}^{I_2} \end{array}\right)$$

make this 0

$$\xrightarrow{2R_1 + R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right)$$

row of zeros on left side

Thus A^{-1} does not exist!

Ex: Find A^{-1} if it exists

when $A = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} & A & & & I_3 & \\ \boxed{3} & 0 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

put a 1 here

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ \boxed{3} & 0 & 3 & 1 & 0 & 0 \\ \boxed{-2} & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

make these 0

$$\xrightarrow[2R_1 + R_3 \to R_3]{-3R_1 + R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & \boxed{-3} & -3 & 1 & -3 & 0 \\ 0 & 5 & 4 & 0 & 2 & 1 \end{array}\right)$$

make this 1

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1/3 & 1 & 0 \\ 0 & \boxed{5} & 4 & 0 & 2 & 1 \end{array}\right)$$

make these 0

$$\xrightarrow[-5R_2+R_3\to R_3]{-R_2+R_1\to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/3 & 0 & 0 \\ 0 & 1 & 1 & -1/3 & 1 & 0 \\ 0 & 0 & \boxed{-1} & 5/3 & -3 & 1 \end{array}\right)$$

make this 1

$$\xrightarrow{-R_3\to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \boxed{1} & 1/3 & 0 & 0 \\ 0 & 1 & \boxed{1} & -1/3 & 1 & 0 \\ 0 & 0 & 1 & -5/3 & 3 & -1 \end{array}\right)$$

make these 0

$$\xrightarrow[-R_3+R_2\to R_2]{-R_3+R_1\to R_1} \left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{}\middle|\underbrace{\begin{matrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{matrix}}_{}\right)$$

I_3 A^{-1}

So when $A = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}$

then $A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$

$$\left[\text{So,} \quad \begin{array}{l} AA^{-1} = I_3 \\ A^{-1}A = I_3 \end{array} \right]$$

Theorem: Let A and B be $n \times n$ matrices that are both invertible [ie, A^{-1} and B^{-1} both exist.]

Then:

① AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

note: $(AB)^{-1} \neq A^{-1}B^{-1}$

② A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

One way to use the inverse is to solve a system.

Let's see how to write a system as a matrix equation.

Consider the following linear system:

$$\boxed{\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}} \qquad (*)$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Then $(*)$ is equivalent to

$$\underbrace{A\vec{x}}_{\text{matrix multiplication}} = \vec{b}$$

Ex: Consider the system

$$\boxed{\begin{aligned} 2x - 3y &= 1 \\ x + 5y &= -2 \end{aligned}} \quad (*)$$

Let

$$A = \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix}, \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \vec{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Then

$$A\vec{x} = \vec{b}$$

becomes

$$\begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

which becomes

$$\begin{pmatrix} (2 \quad -3) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ (1 \quad 5) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

which becomes

$$\begin{pmatrix} 2x - 3y \\ x + 5y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

which is equivalent to

$$\boxed{\begin{aligned} 2x - 3y &= 1 \\ x + 5y &= -2 \end{aligned}} \leftarrow (*)$$