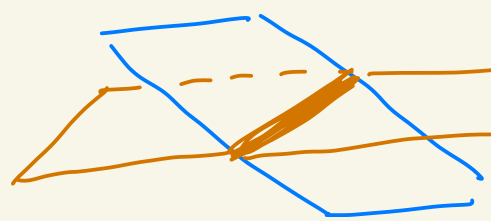
Math 2550-03
2/22/24

Ex continued...

Last time we showed that the solutions to

$$\boxed{\begin{aligned} 5x_1 - 2x_2 + 6x_3 &= 0 \\ -2x_1 + x_2 + 3x_3 &= 1 \end{aligned}} \quad (*)$$



are given by

$$x_1 = 2 - 12t$$
$$x_2 = 5 - 27t$$
$$x_3 = t$$

where t can be any real number

So some example solutions to (*) are illustrated in the following table, but there are infinitely many more, one for each t.

t	$x_1 = 2-12t$	$x_2 = 5-27t$	$x_3 = t$
0	2	5	0
1	-10	-22	1
$1/3$	-2	-4	$1/3$
π	$2-12\pi$	$5-27\pi$	π
⋮	⋮	⋮	⋮

$x_1 = 2$
$x_2 = 5$
$x_3 = 0$
is a sol. to (*)

$x_1 = -10$
$x_2 = -22$
$x_3 = 1$
is another solution

Ex: Solve

$$\boxed{\begin{array}{l} a+3b-2c \qquad\qquad +2e \qquad\quad =0 \\ 2a+6b-5c-2d+4e-3f=-1 \\ \qquad\qquad\quad 5c+10d \qquad +15f=5 \\ 2a+6b \qquad\quad +8d+4e+18f=6 \end{array}} \quad (**)$$

Put this into a matrix and reduce it...

already a 1

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right)$$

make these 0's

$-2R_1+R_2 \to R_2$

$-2R_1+R_4 \to R_4$

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

make this 1

$-R_2 \to R_2$

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & \boxed{5} & 10 & 0 & 15 & 5 \\ 0 & 0 & \boxed{4} & 8 & 0 & 18 & 6 \end{array}\right)$$

make there 0

$-5R_2 + R_3 \to R_3$

$-4R_2 + R_4 \to R_4$

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right)$$

$R_3 \leftrightarrow R_4$

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\frac{1}{6} R_3 \rightarrow R_3$

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

in row echelon form

Back to the land of equations:

$$
\begin{aligned}
a + 3b - 2c + 2e &= 0 \quad ① \\
c + 2d + 3f &= 1 \quad ② \\
f &= 1/3 \quad ③ \\
0 &= 0
\end{aligned}
$$

leading variables: a, c, f

free variables: b, d, e

Now solve for leading variables and give free variables a new name

$$
\begin{aligned}
a &= -3b + 2c - 2e \quad ① \\
c &= 1 - 2d - 3f \quad ② \\
f &= 1/3 \quad ③ \\
b &= s \quad ④ \\
d &= t \quad ⑤ \\
e &= u \quad ⑥
\end{aligned}
$$

Back substitute:

(6) $e = u$

(5) $d = t$

(4) $b = s$

(3) $f = 1/3$

(2) $c = 1 - 2d - 3f = 1 - 2t - 3\left(\frac{1}{3}\right)$

$= -2t$

(1) $a = -3b + 2c - 2e$

$= -3s + 2(-2t) - 2u$

$= -3s - 4t - 2u$

Answer

$a = -3s - 4t - 2u$

$b = s$

$c = -2t$

$d = t$

$e = u$

$f = 1/3$

where s, t, u can be any real numbers

There are an infinite # of solutions. For example, one of them is:

$$\left.\begin{aligned} a &= -3(1)-4(1)-2(1) = -9 \\ b &= 1 \\ c &= -2(1) = -2 \\ d &= 1 \\ e &= 1 \\ f &= 1/3 \end{aligned}\right\} \begin{aligned} &\text{when} \\ &s=1 \\ &t=1 \\ &u=1 \end{aligned}$$

<u>Theorem:</u> A system of linear equations either has

(i) no solutions,

(ii) exactly one solution,

or (iii) infinitely many solutions.

Topic 4 – Inverse of a matrix

Motivation: Non-zero numbers have multiplicative inverses.

For example, $2 \cdot \frac{1}{2} = 1$

2 and $\frac{1}{2}$ are multiplicative inverses

$$2^{-1} = \frac{1}{2}$$

Can we do this for matrices?

Recall: $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Def: Let A be an $n \times n$ matrix. [So A is a square matrix]

We say that A is <u>invertible</u> if there exists an $n \times n$ matrix B where

$$AB = I_n = BA$$

$AB = I_n$ and $BA = I_n$

If $AB = I_n = BA$, then we say that A and B are <u>inverses</u> of each other.

Ex: Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

and $B = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$.

Let's show that A and B are inverses of each other.

We have that

$$AB = \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}}_{2\times 2}$$

✓

answer is 2×2

$$= \begin{pmatrix} (1 \;\; 1)\cdot\begin{pmatrix} -1 \\ 2 \end{pmatrix} & (1 \;\; 1)\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ & \\ (2 \;\; 1)\cdot\begin{pmatrix} -1 \\ 2 \end{pmatrix} & (2 \;\; 1)\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} (1)(-1)+(1)(2) & (1)(1)+(1)(-1) \\ (2)(-1)+(1)(2) & (2)(1)+(1)(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

And

$$BA = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (-1 \;\; 1) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} & (-1 \;\; 1) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ (2 \;\; -1) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} & (2 \;\; -1) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} (-1)(1)+(1)(2) & (-1)(1)+(1)(1) \\ (2)(1)+(-1)(2) & (2)(1)+(-1)(1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Since $AB = I_2$ and $BA = I_2$, we know A and B are inverses of each other.
