Math 2550-03
2/15/24

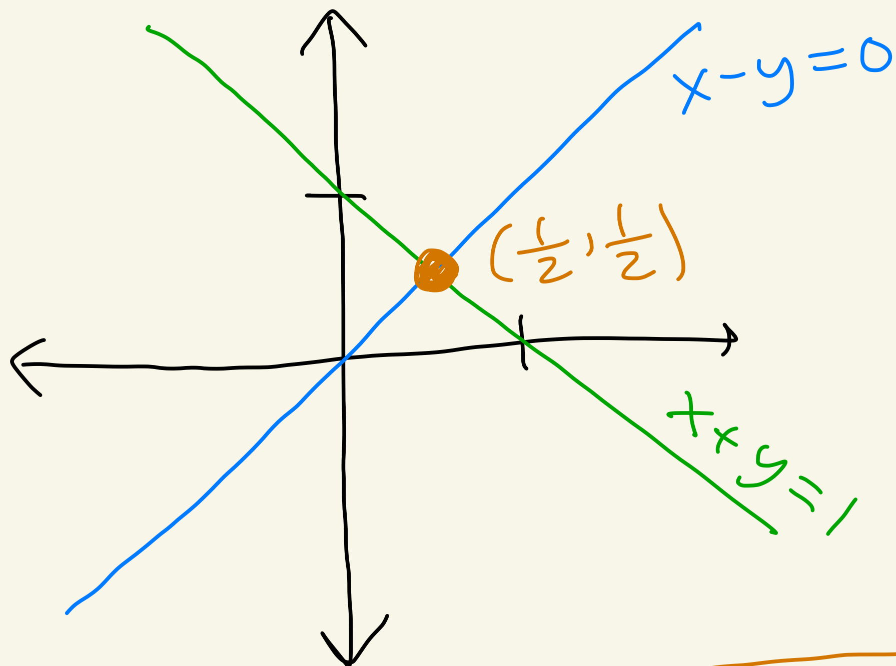
Theorem: Applying an elementary row operation to a system of linear equations does not change the solution space to the system.

Ex:

system

$x + y = 1$

$x - y = 0$



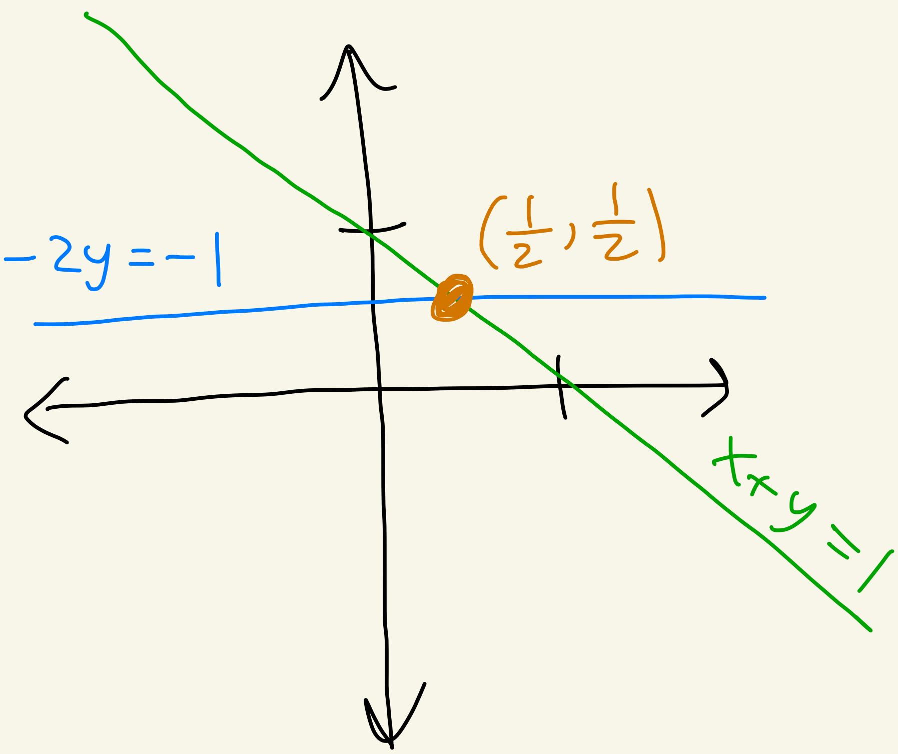
$\boxed{\text{The solution space is } (x,y) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)}$

Let's apply an elementary row operation and see what happens.

$$\overbrace{\begin{matrix} x+y=1 \\ x-y=0 \end{matrix}}\qquad\qquad \overbrace{\begin{matrix} x+y=1 \\ -2y=-1 \end{matrix}}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{-R_1+R_2\to R_2} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & -1 \end{array}\right)$$

$$\boxed{\begin{array}{rl} \left(\begin{array}{cc|c} -1 & -1 & -1 \end{array}\right) & \leftarrow \boxed{-R_1} \\ +\left(\begin{array}{cc|c} 1 & -1 & 0 \end{array}\right) & \leftarrow \boxed{R_2} \\ \hline \left(\begin{array}{cc|c} 0 & -2 & -1 \end{array}\right) & \leftarrow \boxed{\text{new } R_2} \end{array}}$$



We get the same solution space of $(x,y)=\left(\frac{1}{2},\frac{1}{2}\right)$

Def: If a row of a matrix does not consist entirely of zeros then the **leading entry** in that row is the first non-zero entry when scanning from left to right.

Ex:

$$\left(\begin{array}{cccc} 1 & 0 & 2 & -1 \\ 0 & 0 & 3 & 1 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \leftarrow \text{row } 3 \\ \leftarrow \text{row } 4 \end{array}$$

leading entry of row 1 is 1
leading entry of row 2 is 3
leading entry of row 3 is -2
row 4 has no leading entry.

Def: A matrix is in **row echelon form** if three conditions are true:

1. If there are any rows that consist entirely of zeros, then those rows are grouped together at the bottom of the matrix.
2. In any two consecutive rows that do not consist entirely of zeros, the leading entry in the lower row is to the right of the leading entry in the upper row.
3. If a row does not consist entirely of zeros then the leading entry is 1.

Ex:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

(1) ✓
(2) ✓
(3) ✗

(leading entries circled)

Not in row echelon form

Ex:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1/4 \end{pmatrix}$$

(1) ✓
(2) ✓
(3) ✓

Is in row echelon form

Ex:

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

rows of zeros

(1) ✓
(2) ✓
(3) ✓

matrix is in row echelon form

Ex:

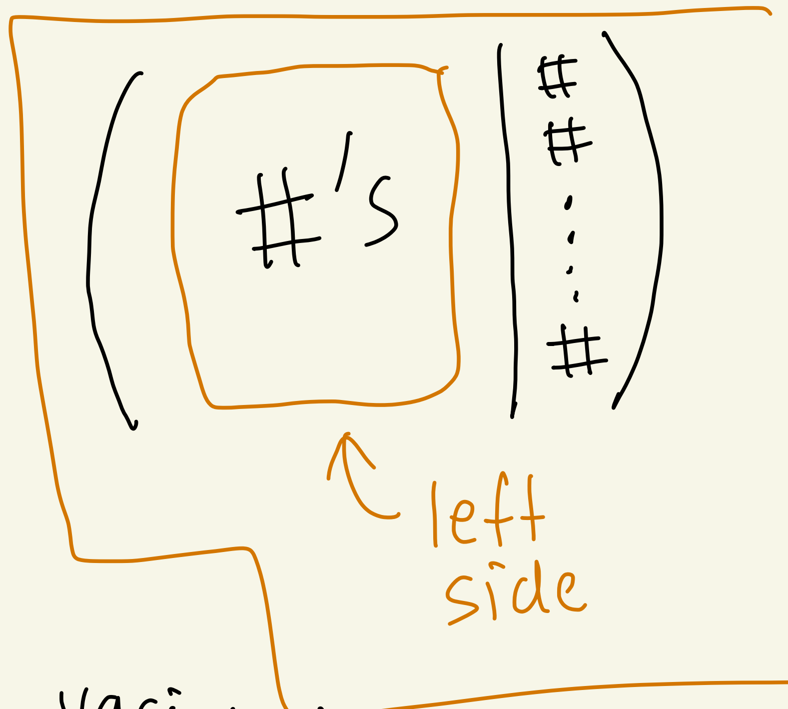
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 & -1 \end{pmatrix}$$

1 ✗
2 ✗
3 ✓

Matrix is not in row echelon form

Def: Suppose you have an augmented matrix for a system of linear equations.

Suppose you use elementary row operations to put the left side of the matrix into row echelon form.



The variable corresponding to the leading entry of a row is called a leading variable (or pivot variable).

Any variable that doesn't occur as a leading variable is called a free variable.

Ex:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

left side is in row echelon form

leading entries are circled

Suppose the above matrix corresponds to this system:

$$\boxed{\begin{aligned} x \quad\quad + z &= 3 \\ y + z &= 2 \\ 0 &= 0 \end{aligned}}$$

leading variables are x, y

free variable is z

How to preform Gaussian elimination

① Put your system into an augmented matrix.

② Use elementary row operations to put the left side of the augmented matrix into row echelon form.

③ Write down the new system corresponding to the matrix from step 2.

④

case (a): If one of the equations in the system from step 3 is $0 = c$ where $c \neq 0$, then the system has no solutions.

case (b): If case (a) doesn't occur, then we use back substitution to solve the system as follows:

(i) Solve the equations for the leading variables.

(ii) Assign each free variable a new name as it can take any value.

(iii) Starting with the bottom / last equation and working upwards, successively substitute the each equation into the equation above it.