Math 2550
2/13/24

Topic 3 – Systems of linear equations

Def: A linear equation in the n variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b} \quad (*)$$

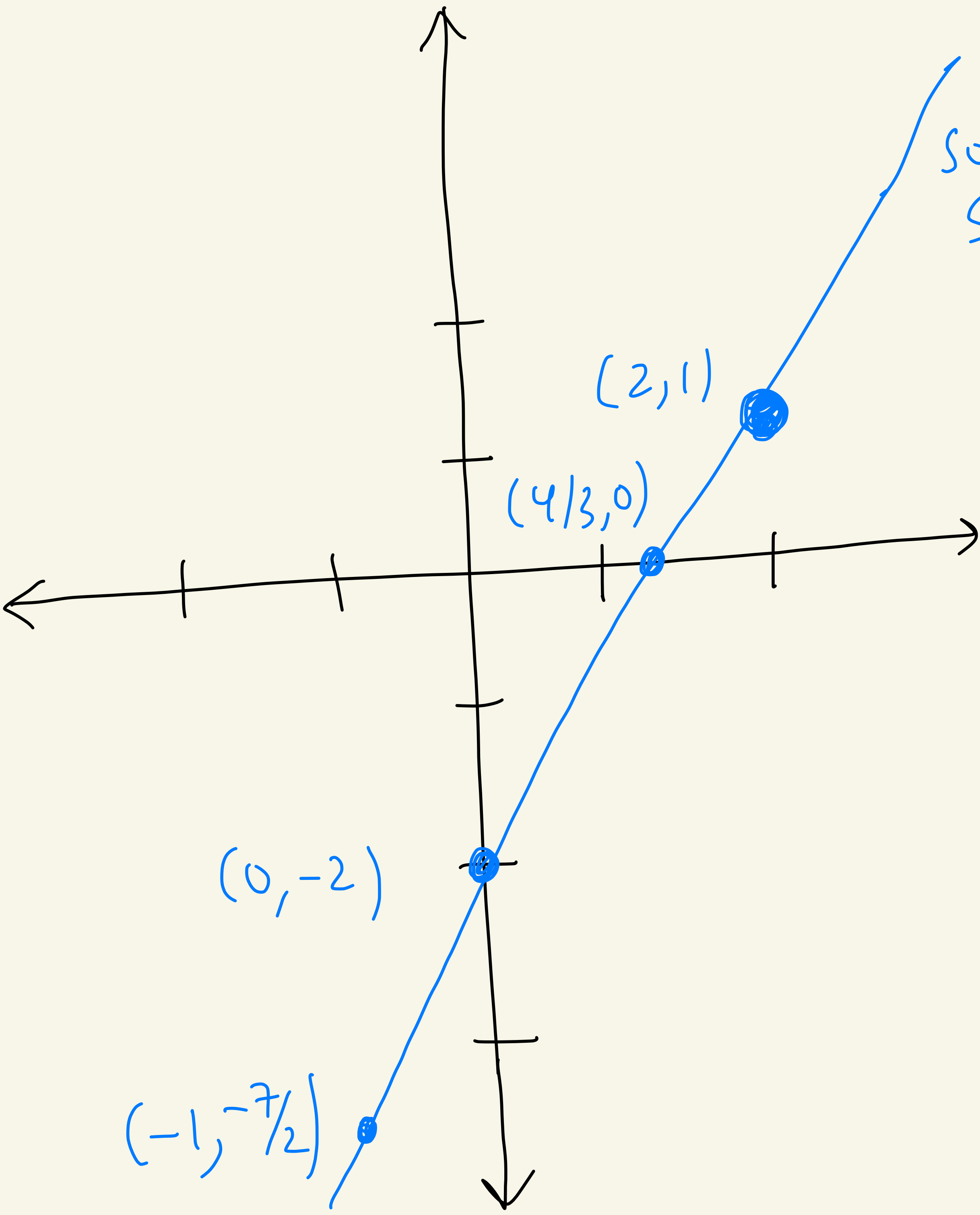
where $a_1, a_2, \ldots, a_n, b$ are constant real numbers.

The solution space of $(*)$ consists of all $(x_1, x_2, \ldots, x_n)$ that solve $(*)$.

Ex:

$$\underbrace{3x - 2y = 4}_{a_1x_1 + a_2x_2 = b}$$

linear eqn. w/ $n=2$ variables/unknowns



solution space is all (x,y) on this graph

Examples of linear equations:

$$10x - 2y + \frac{1}{3}z = 5$$

$$\sqrt{2}\,x_1 - \pi x_2 + x_3 + 2x_4 = 0$$

Not linear equations:

$$2x^2 + y = 7$$

$$5\cos(x) + z = 0$$

Def: A <u>system of m linear equations in the n unknowns $x_1, x_2, \ldots, x_n$</u> is a list of m equations of the form:

$$
\boxed{\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \quad \vdots \qquad \qquad \quad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}} \qquad (*)
$$

where the a_{ij} are constants.

The <u>augmented matrix</u> for $(*)$ is

$$
\left(\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right)
$$

represents the = sign

x_1 column $\quad$ x_2 column $\quad$ x_n column

The solution space of (*) consists of all $(x_1, x_2, \ldots, x_n)$ that simultaneously solve all m equations in (*), ie

The common solutions to all m equations.

Ex:

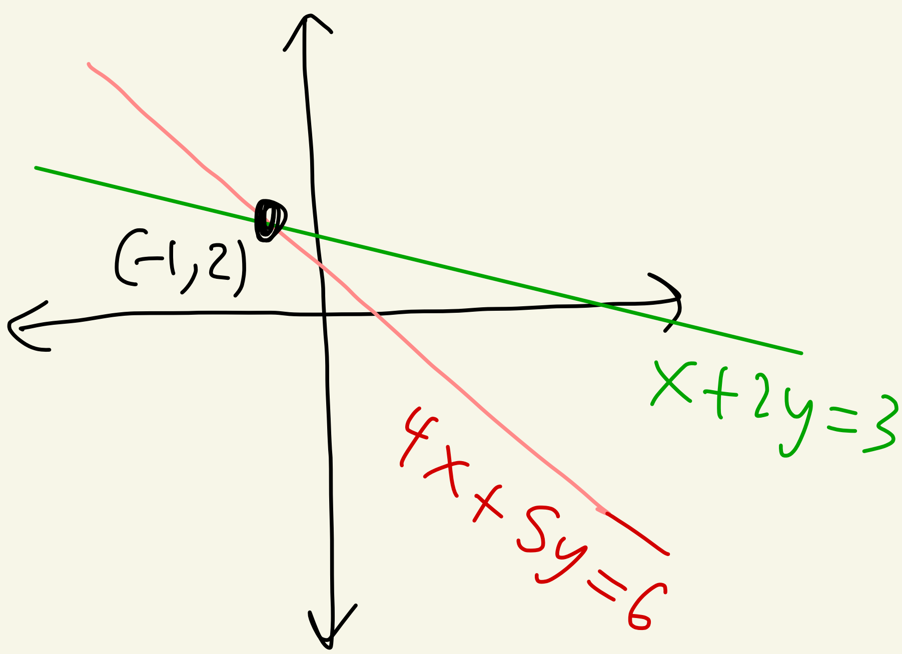
system:

$$\boxed{\begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned}}$$

m=2 lin eqs.
n=2 unknowns

augmented matrix

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right)$$



sol. space for system is $(x,y)=(-1,2)$

Ex:

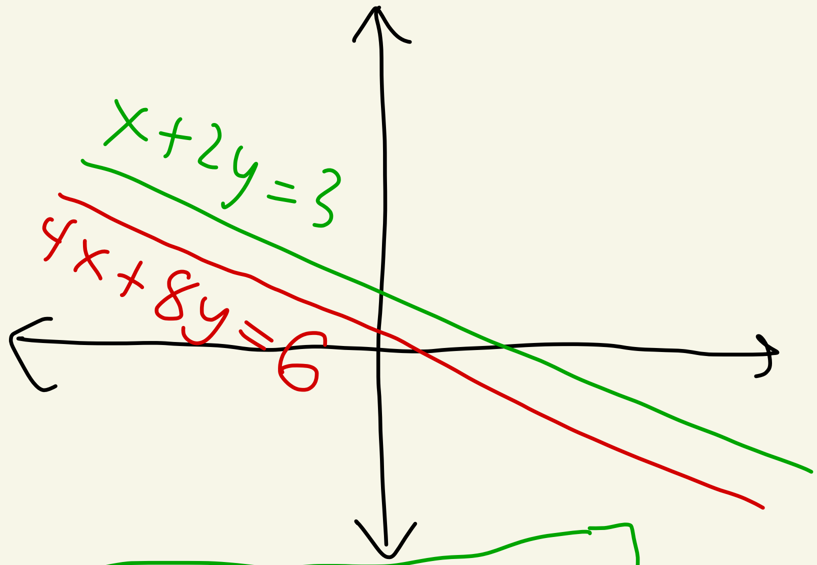
<u>system</u>

$$\begin{aligned} x+2y&=3 \\ 4x+8y&=6 \end{aligned}$$

$m=2$ lin. eqns
$n=2$ unknowns

<u>augmented matrix</u>

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 6 \end{array}\right)$$



parallel lines
solution space
is empty.
No common solutions

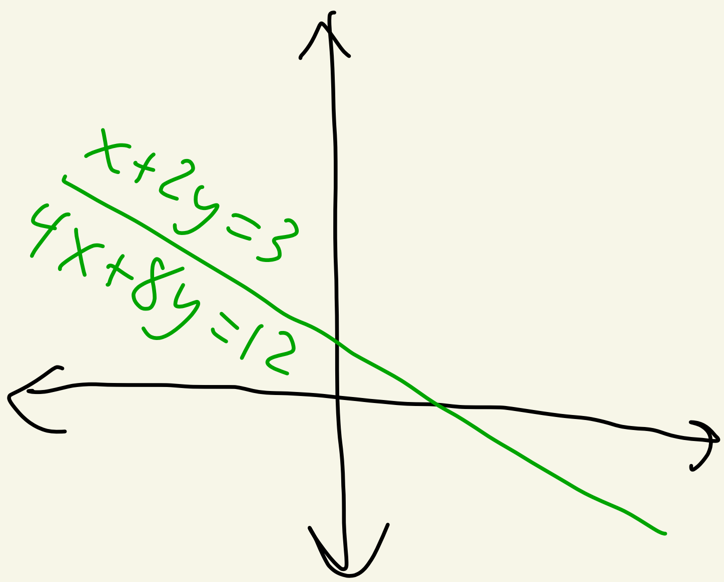
Ex:

<u>System</u>

$$\begin{aligned} x+2y&=3 \\ 4x+8y&=12 \end{aligned}$$

<u>augmented matrix</u>

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 12 \end{array}\right)$$



the solution space
is infinite. It's
the entire line.

Ex:

$$\boxed{\begin{aligned} x + y + 2z &= 9 \\ 2x \phantom{{}+6y} - 3z &= 1 \\ -x + 6y - 5z &= 0 \end{aligned}}$$

$m = 3$ lin. eqns

$n = 3$ unknowns

augmented matrix $\rightarrow$ $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 0 & -3 & 1 \\ -1 & 6 & -5 & 0 \end{array}\right)$

Ex:

$$\boxed{\begin{aligned} x - y \phantom{{}+z-17w} &= 3 \\ y + z - 17w &= 1 \\ 2x + y - z + w &= 0 \end{aligned}}$$

augmented matrix $\rightarrow$ $\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 3 \\ 0 & 1 & 1 & -17 & 1 \\ 2 & 1 & -1 & 1 & 0 \end{array}\right)$

Def: Given a system of linear equations there are three operations that we call elementary row operations

They are:

1. Multiply one of the rows/equations by a non-zero constant
2. Interchange two rows/equations
3. Add a multiple of one row/equation to another row/equation

Ex: (Multiply a row/equation by a non-zero constant)

equation viewpoint

$$\boxed{\begin{aligned} 2x - 4y + 6z &= 3 \\ x \qquad + z &= 1 \\ 10x - y - z &= 0 \end{aligned}} \xrightarrow{\frac{1}{2}R_1 \to R_1} \boxed{\begin{aligned} x - 2y + 3z &= \tfrac{3}{2} \\ x \qquad + z &= 1 \\ 10x - y - z &= 0 \end{aligned}}$$

matrix viewpoint

$$\left(\begin{array}{ccc|c} 2 & -4 & 6 & 3 \\ 1 & 0 & 1 & 1 \\ 10 & -1 & -1 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 3/2 \\ 1 & 0 & 1 & 1 \\ 10 & -1 & -1 & 0 \end{array}\right)$$

Ex: (Interchange two rows / equations)

equation viewpoint

$$\begin{aligned} 2x - 4y + 6z &= 3 \\ x \qquad\quad + z &= 1 \\ 10x - y - z &= 0 \end{aligned} \xrightarrow{R_1 \leftrightarrow R_2} \begin{aligned} x \qquad\quad + z &= 1 \\ 2x - 4y + 6z &= 3 \\ 10x - y - z &= 0 \end{aligned}$$

matrix viewpoint

$$\left(\begin{array}{ccc|c} 2 & -4 & 6 & 3 \\ 1 & 0 & 1 & 1 \\ 10 & -1 & -1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & -4 & 6 & 3 \\ 10 & -1 & -1 & 0 \end{array}\right)$$

Ex: (Add a multiple of one row/eqn to another row/eqn)

equation viewpoint

$$\begin{array}{rcl} x - y + z &=& 1 \\ 2x + y - z &=& 0 \\ y + z &=& 4 \end{array} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{array}{rcl} x - y + z &=& 1 \\ 3y - 3z &=& -2 \\ y + z &=& 4 \end{array}$$

$$\begin{array}{rl} -2x + 2y - 2z = -2 & \leftarrow -2R_1 \\ + \,(2x + y - z = 0) & \leftarrow R_2 \\ \hline 3y - 3z = -2 & \leftarrow \text{new } R_2 \end{array}$$

matrix viewpoint

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 4 \end{array}\right) \xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & -2 \\ 0 & 1 & 1 & 4 \end{array}\right)$$

$$\begin{array}{rl} \left(\begin{array}{ccc|c} -2 & 2 & -2 & -2 \end{array}\right) & \leftarrow -2R_1 \\ + \left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \end{array}\right) & \leftarrow R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 0 & 3 & -3 & -2 \end{array}\right) \leftarrow \text{new } R_2$$