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Def: Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$. Suppose

$$\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$$

and $\vec{w} = \langle b_1, b_2, \ldots, b_n \rangle$.

The dot product of $\vec{v}$ and $\vec{w}$ is defined be

$$\vec{v} \cdot \vec{w} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

So, the dot product of two vectors is a number

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle 5, 6 \rangle$ and $\vec{w} = \langle -1, 0 \rangle$.

Then,

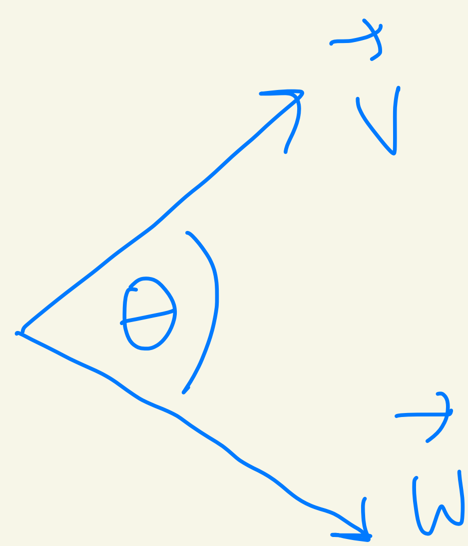
$$\vec{v} \cdot \vec{w} = \langle 5, 6 \rangle \cdot \langle -1, 0 \rangle$$

$$= (5)(-1) + (6)(0)$$

$$= -5$$

Recall from Calculus: In $\mathbb{R}^2$ or $\mathbb{R}^3$ we have:

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}$$



Ex: In $\mathbb{R}^6$ we have that

$$\langle 1, 0, 2, 4, -1, -2 \rangle \cdot \langle 2, 10, 3, 0, 5, 0 \rangle$$

$$= (1)(2) + (0)(10) + (2)(3) + (4)(0) + (-1)(5) + (-2)(0)$$

$$= 2 + 0 + 6 + 0 - 5 + 0$$

$$= 3$$

Properties of the dot product

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$
and α be a scalar in $\mathbb{R}$.

Then:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $\underbrace{\alpha(\vec{u} \cdot \vec{v}) = (\alpha \vec{u}) \cdot \vec{v} = \vec{u} \cdot (\alpha \vec{v})}$

$5(\vec{u} \cdot \vec{v}) = (5\vec{u}) \cdot \vec{v} = \vec{u} \cdot (5\vec{v})$

if $\alpha = 5$

$\boxed{\text{Let's prove (2) when } n=3}$:

Let $\vec{u}, \vec{v}, \vec{w}$ be in $\mathbb{R}^3$.

We must show that

$$\boxed{\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}}$$

We know that

$$\vec{u} = \langle a, b, c \rangle$$

$$\vec{v} = \langle d, e, f \rangle$$

and $\vec{w} = \langle g, h, i \rangle$

where $a, b, c, d, e, f, g, h, i \in \mathbb{R}$.

Then,

$\vec{u} \cdot (\vec{v} + \vec{w})$

$= \langle a, b, c \rangle \cdot \left(\langle d, e, f \rangle + \langle g, h, i \rangle \right)$

$$= \langle a, b, c \rangle \cdot \langle d+g, e+h, f+i \rangle$$

$$= a(d+g) + b(e+h) + c(f+i)$$

$$= ad + ag + be + bh + cf + ci \quad (*)$$

Also,

$$\vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$= \langle a, b, c \rangle \cdot \langle d, e, f \rangle$$

$$+ \langle a, b, c \rangle \cdot \langle g, h, i \rangle$$

$$\left.\begin{aligned} &= ad + be + cf \\ &\quad + ag + bh + ci \end{aligned}\right\} (**)$$

We see that $(*) = (**)$

and so $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$.

$\blacksquare$

Hw 1 - Part 1

10. List 3 elements from the set

$$S = \{ c_1\langle 1,1,1\rangle + c_2\langle 0,0,5\rangle \mid c_1, c_2 \in \mathbb{R} \}$$

read: S consists of all vectors of the form $c_1\langle 1,1,1\rangle + c_2\langle 0,0,5\rangle$ where c_1, c_2 are real #s.

If $c_1 = 4$ and $c_2 = 1$, then we get

$$4 \cdot \langle 1,1,1 \rangle + 1 \cdot \langle 0,0,5 \rangle$$

$$= \langle 4,4,4 \rangle + \langle 0,0,5 \rangle$$

$$= \langle 4,4,9 \rangle$$

So, $\langle 4,4,9 \rangle$ is in the set S.

If $c_1 = 1$ and $c_2 = 0$, then we get

$$1 \cdot \langle 1,1,1 \rangle + 0 \cdot \langle 0,0,5 \rangle$$

$$= \langle 1,1,1 \rangle + \langle 0,0,0 \rangle$$

$$= \langle 1,1,1 \rangle$$

So, $\langle 1,1,1 \rangle$ is in the set S

If $c_1 = -1$ and $c_2 = 1$,

then we get

$$(-1)\langle 1,1,1\rangle + 1\cdot\langle 0,0,5\rangle$$

$$= \langle -1,-1,-1\rangle + \langle 0,0,5\rangle$$

$$= \langle -1,-1,4\rangle$$

So, $\langle -1,-1,4\rangle \in S$

So,

$$S = \{\langle 4,4,9\rangle, \langle 1,1,1\rangle, \langle -1,-1,4\rangle, \ldots\}$$

infinitely many more

Topic 2 – Matrices

<u>Def:</u> A matrix is a rectangular array of numbers. If M is a matrix with m rows and n columns then we say that M is an $\underbrace{m \times n}_{\text{read: "m by n"}}$ matrix.

Abstractly you can write an $m \times n$ matrix M as follows:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where a_{ij} the entry in row i and column j.

Ex:

$$M = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

M is $\underbrace{2}_{\boxed{\text{2 rows}}} \times \underbrace{3}_{\boxed{\text{3 columns}}}$

$$\boxed{\begin{aligned} a_{11} &= 1 \\ a_{12} &= -1 \\ a_{13} &= 0 \\ a_{21} &= 3 \\ a_{22} &= 2 \\ a_{23} &= 1 \end{aligned}}$$

Ex:

$$M = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{pmatrix}$$

M is 1×4

$a_{11} = 1$ $\quad a_{13} = 3$

$a_{12} = 2$ $\quad a_{14} = 4$

You could use commas:

$$M = (1, 2, 3, 4)$$

<u>Note:</u> Sometimes we want to think of a vector as a matrix.

Let $\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$.

We can think of $\vec{v}$ as an $n \times 1$ matrix $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

or we can think of $\vec{v}$ as a $1 \times n$ matrix $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$

<u>Ex:</u> $\vec{v} = \langle 1, 2, 3 \rangle$

$\underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{\vec{v} \text{ as a } 3\times 1 \text{ matrix}}$ $\qquad$ $\underbrace{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}}_{\vec{v} \text{ as a } 1\times 3 \text{ matrix}}$

<u>Def:</u> Let A and B be $m \times n$ matrices. [So, A and B have the same size]

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

Define $A+B$ to be the following $m \times n$ matrix:

$$A+B=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

Define $A-B$ to be the following $m\times n$ matrix:

$$A-B=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \cdots & a_{2n}-b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \cdots & a_{mn}-b_{mn} \end{pmatrix}$$

If α is a scalar in $\mathbb{R}$, define αA to be the following $m\times n$ matrix:

$$\alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}$$