Math 2550-03

1/30/24

Topic 1 – Vectors

<u>Def:</u> Let $n \geq 1$ be an integer.

[So n can be $1, 2, 3, 4, 5, \ldots$]

An <u>n-dimensional real vector</u> is a list of n real numbers.

We use brackets $<$ and $>$ for vectors.

We use an arrow over a variable that is a vector such as $\vec{v}$.

Ex: Some 2-dimensional vectors:

$\langle 5, -1 \rangle$

$\langle 4, 7 \rangle$

$\langle 7, 4 \rangle$

different vectors (order matters)

$\langle 2, 2 \rangle$

Ex: Some 3-dimensional vectors:

$\langle 1, -2, \pi \rangle$

$\langle 6, 7, 10 \rangle$

Ex: Some 6-dimensional vectors:

$\langle 1, 2, 3, 4, 5, 6 \rangle$

$\langle 0, 1, -1, 2, 3, 4 \rangle$

Def: Let $n \geq 1$ be an integer.
[So, n can be $1, 2, 3, 4, \ldots$]

Define $\mathbb{R}^n$ to be the set of all n-dimensional real vectors.

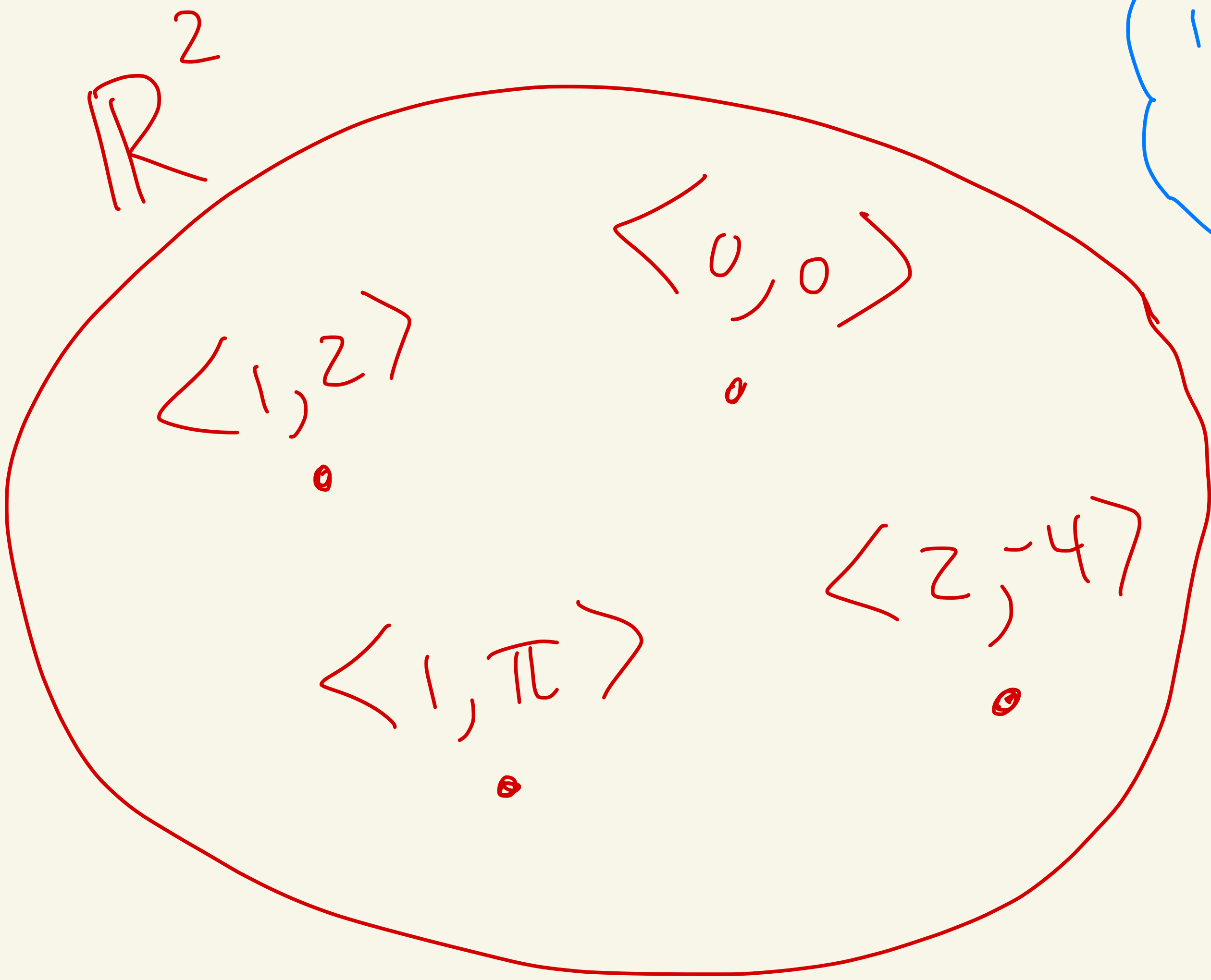
That is,

$$\mathbb{R}^n = \underbrace{\left\{ \langle a_1, a_2, \ldots, a_n \rangle \;\middle|\; a_1, a_2, \ldots, a_n \in \mathbb{R} \right\}}_{\text{read: the set of all } \langle a_1, a_2, \ldots, a_n \rangle \text{ where } a_1, a_2, \ldots, a_n \text{ are real \#'s.}}$$

Ex: $\mathbb{R}^2 = \{\langle a_1, a_2 \rangle \mid a_1, a_2 \in \mathbb{R}\}$

$$= \{\langle 1,2 \rangle, \langle 0,0 \rangle, \ldots\}$$

infinitely many more

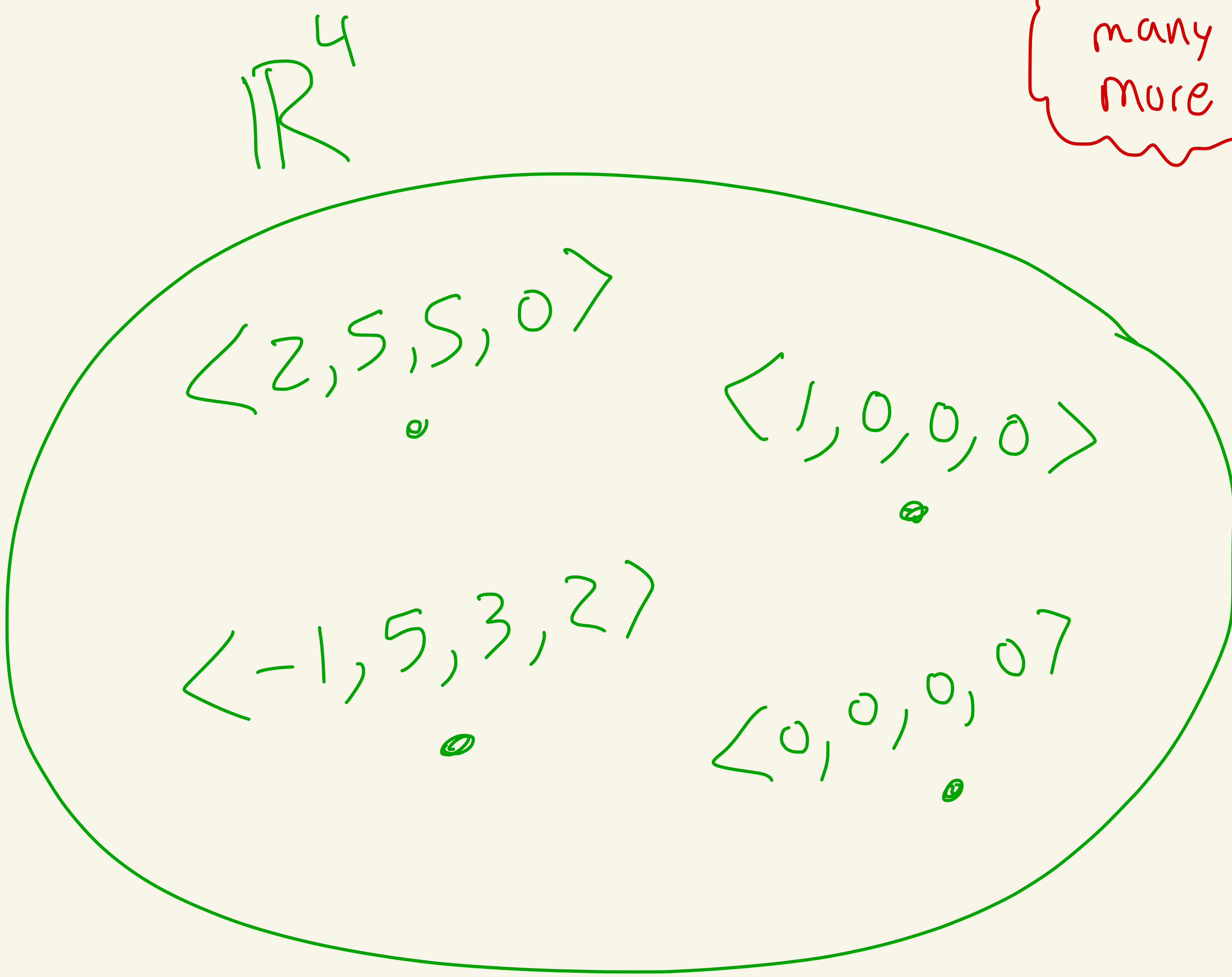


Ex:

$$\mathbb{R}^4 = \{ \langle a_1, a_2, a_3, a_4 \rangle \mid a_1, a_2, a_3, a_4 \in \mathbb{R} \}$$

$$= \{ \langle 2,5,5,0 \rangle, \langle 1,0,0,0 \rangle, \ldots \}$$

infinitely many more



Def: The length (or norm or magnitude) of a vector $\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ in $\mathbb{R}^n$ is

$$\|\vec{v}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

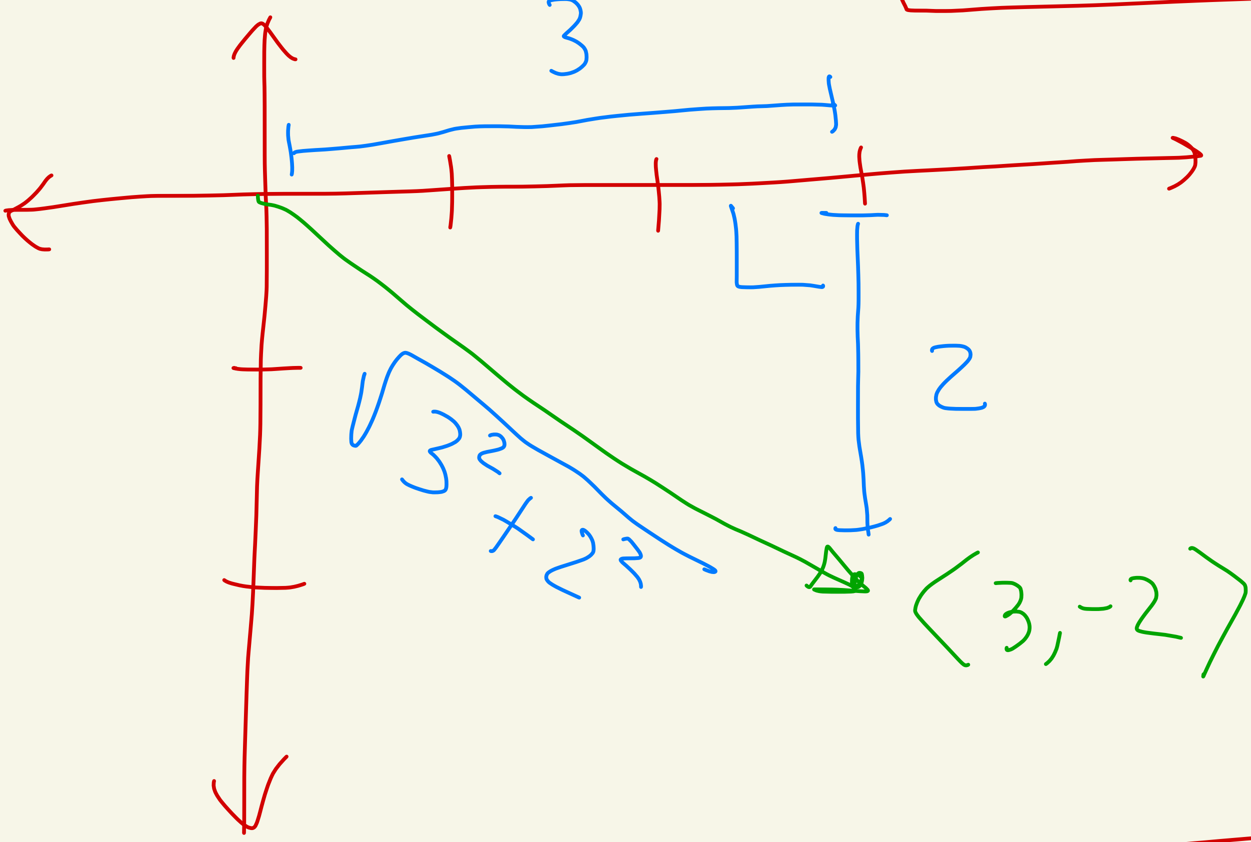
$$\boxed{\text{Some people write } |\vec{v}| \text{ instead of } \|\vec{v}\|}$$

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle 3, -2 \rangle$.

Then,

$$\|\vec{v}\| = \sqrt{3^2 + (-2)^2} = \sqrt{13} \approx 3.6$$

Formula comes from Pythagorean formula



Ex: In $\mathbb{R}^8$, let

$$\vec{v} = \langle -1, 0, 2, 0, 0, 3, 4, -2 \rangle$$

Then,

$$\|\vec{v}\| = \sqrt{(-1)^2 + 0^2 + 2^2 + 0^2 + 0^2 + 3^2 + 4^2 + (-2)^2}$$

$$= \sqrt{1 + 4 + 9 + 16 + 4}$$

$$= \sqrt{34}$$

$$\approx 5.8309\ldots$$

Operations on vectors:

Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$ and α be a $\underbrace{\text{scalar}}_{\text{number}}$ in $\mathbb{R}$.

Some greek letters
α - alpha
β - beta
γ - gamma
δ - delta
ω - omega
σ - sigma

Suppose

$$\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$$

$$\vec{w} = \langle b_1, b_2, \ldots, b_n \rangle$$

Define vector addition as

$$\vec{v} + \vec{w} = \langle a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n \rangle$$

Define vector subtraction as

$$\vec{v} - \vec{w} = \langle a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n \rangle$$

Define vector scaling as

$$\alpha \vec{v} = \langle \alpha a_1, \alpha a_2, \ldots, \alpha a_n \rangle$$

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle 2, -1 \rangle$ and $\vec{w} = \langle -3, 2 \rangle$.

Then,

$$\vec{v} + \vec{w} = \langle 2, -1 \rangle + \langle -3, 2 \rangle$$

$$= \langle 2 + (-3), -1 + 2 \rangle$$

$$= \langle -1, 1 \rangle$$

$$\vec{w} - \vec{v} = \langle -3, 2 \rangle - \langle 2, -1 \rangle$$

$$= \langle -3 - 2, 2 - (-1) \rangle$$

$$= \langle -5, 3 \rangle$$

$$\underbrace{(-10)}_{\alpha}\vec{w} = (-10)\langle -3, 2\rangle$$

$$= \langle (-10)(-3), (-10)(2)\rangle$$

$$= \langle 30, -20\rangle$$

<u>Ex:</u> In $\mathbb{R}^5$ we have:

$$\langle 2, 0, 0, 1, -3\rangle + \langle 2, 5, 5, 0, 0\rangle$$

$$= \langle 2+2, 0+5, 0+5, 1+0, -3+0\rangle$$

$$= \langle 4, 5, 5, 1, -3\rangle$$

$$5\langle 2, 0, 0, 1, -3\rangle = \langle 10, 0, 0, 5, -15\rangle$$

$\boxed{\text{Notation:}}$ In $\mathbb{R}^n$, the <u>zero vector</u>, notated by $\vec{0}$, is the vector with all zeros in it.

In $\mathbb{R}^2$, $\vec{0} = \langle 0, 0 \rangle$.

In $\mathbb{R}^3$, $\vec{0} = \langle 0, 0, 0 \rangle$

In $\mathbb{R}^4$, $\vec{0} = \langle 0, 0, 0, 0 \rangle$

And so on...

Properties of vectors:

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$

and let α, β be $\underbrace{\text{scalars}}_{\text{numbers}}$ in $\mathbb{R}$.

Then:

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ ← (commutative property)
2. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ ← (associative property)
3. $\alpha(\beta \vec{u}) = (\alpha\beta)\vec{u}$ ← Ex: $2(5\vec{u}) = 10\vec{u}$
4. $(\alpha + \beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$ ← Ex: $5\vec{u} = (3+2)\vec{u} = 3\vec{u} + 2\vec{u}$
5. $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$ ← Ex: $5(\vec{u} + \vec{v}) = 5\vec{u} + 5\vec{v}$
6. $\vec{u} + \vec{0} = \vec{u}$
   $\vec{0} + \vec{u} = \vec{u}$
7. $\vec{u} + (-\vec{u}) = \vec{0}$
   $(-\vec{u}) + \vec{u} = \vec{0}$

$\boxed{\text{proof of (3) when } n=2}$:

Let $\vec{u}$ be a vector in $\mathbb{R}^2$ and let α, β be scalars in $\mathbb{R}$.

Then $\vec{u} = \langle a_1, a_2 \rangle$ where a_1, a_2 are real numbers.

Then,

$$\alpha(\beta\vec{u}) = \alpha(\beta\langle a_1, a_2\rangle)$$
$$= \alpha\langle \beta a_1, \beta a_2\rangle$$
$$= \langle \alpha\beta a_1, \alpha\beta a_2\rangle$$
$$= \alpha\beta\langle a_1, a_2\rangle$$
$$= \alpha\beta\vec{u}$$

end of proof

$\blacksquare$