Math 2550-01

4/9/24

Test 2 is in two weeks. Tuesday 4/23.

Review day is Thursday 4/18.

Special example:

The "trivial" vector space is

$$V = \{ \vec{0} \}$$

the vector space with just one vector

It turns out that V has no basis.

We just define its dimension to be 0.

Ex: Let $V = \mathbb{R}^n$ and $F = \mathbb{R}$.

The standard basis for $\mathbb{R}^n$ is $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ where $\vec{v}_i$ has a 1 in spot i and 0's everywhere else. This gives $\dim(\mathbb{R}^n) = n$.

n	standard basis for $\mathbb{R}^n$	$\dim(\mathbb{R}^n)$
2	$\vec{v}_1 = \langle 1,0 \rangle$, $\vec{v}_2 = \langle 0,1 \rangle$	2
3	$\vec{v}_1 = \langle 1,0,0 \rangle$, $\vec{v}_2 = \langle 0,1,0 \rangle$ $\vec{v}_3 = \langle 0,0,1 \rangle$	3
4	$\vec{v}_1 = \langle 1,0,0,0 \rangle$ $\vec{v}_2 = \langle 0,1,0,0 \rangle$	4

	$\vec{v}_3 = \langle 0, 0, 1, 0 \rangle$ $\vec{v}_4 = \langle 0, 0, 0, 1 \rangle$		
$\vdots$	$\vdots$		$\vdots$

<u>Ex:</u> Let n be an integer with $n \geq 0$. Then

$$P_n = \{ a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \mid a_0, \dots, a_n \text{ are in } \mathbb{R} \}$$

set of polynomials of degree $\leq n$

The standard basis for P_n is

$$\left.\begin{aligned} \vec{v}_0 &= 1 \\ \vec{v}_1 &= x \\ \vec{v}_2 &= x^2 \\ &\vdots \\ \vec{v}_n &= x^n \end{aligned}\right\} \text{ n+1 vectors}$$

So, $\dim(P_n) = n+1$

n	standard basis for P_n	$\dim(P_n)$
0	1	1
1	$1, x$	2

2	$1, x, x^2$	3
3	$1, x, x^2, x^3$	4
$\vdots$	$\vdots$	$\vdots$

Ex:

$$5 + x - x^3 = 5 \cdot 1 + 1 \cdot x + 0 \cdot x^2 - 1 \cdot x^3$$

Ex: Let $V = M_{2,2}$ ← set of all 2x2 matrices

$F = \mathbb{R}$.

Note:

$$\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (-1) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 5 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

One can show [HW 7 - Part 1]

that

$\vec{v}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

$\vec{v}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\vec{v}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

is a basis for $M_{2,2}$.

Thus, $\dim(M_{2,2}) = 4$

<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F with $\dim(V) = n$.

So, V has a basis with n vectors

Let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ be vectors from V.

1. If $m < n$, then $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ do not span V.

2. If $m > n$, then $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ are linearly dependent.

Ex: Let $V = \mathbb{R}^3$, $F = \mathbb{R}$.

Then, $\dim(\mathbb{R}^3) = 3 \leftarrow \boxed{n=3}$

Let

$$\left.\begin{aligned}\vec{w}_1 &= \langle 1, 5, 0 \rangle \\ \vec{w}_2 &= \langle 2, -1, 0 \rangle\end{aligned}\right\} \quad m = 2$$

① from above says since we ~~only~~ only have 2 vectors $\vec{w}_1, \vec{w}_2$ in a 3-dimensional space, then $\vec{w}_1, \vec{w}_2$ don't span all of $\mathbb{R}^3$. Ie you can't make all the vectors in $\mathbb{R}^3$ from just $\vec{w}_1$ and $\vec{w}_2$.

Ex: Let $V = P_2$, $F = \mathbb{R}$.

We know that $\underbrace{\dim(P_2) = 3}_{\text{basis: } 1,\, x,\, x^2}$

Let

$$\left.\begin{aligned} \vec{w}_1 &= 1 + x \\ \vec{w}_2 &= 1 - x^2 \\ \vec{w}_3 &= x \\ \vec{w}_4 &= x^2 \end{aligned}\right\} \begin{array}{l} m = 4 \text{ vectors} \\ \text{in a } n = 3 \\ \text{dimensional} \\ \text{space} \end{array}$$

(2) from above says since we have 4 vectors in a 3-dimensional space [$4 > 3$] then $\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4$ must be linearly dependent.

Here we have

$$\underbrace{1\cdot\vec{w}_1 - 1\cdot\vec{w}_2 - \vec{w}_3 - \vec{w}_4}_{1+x-1+x^2-x-x^2=\vec{0}} = \vec{0}$$

$$\vec{w}_1 = \vec{w}_2 + \vec{w}_3 + \vec{w}_4$$

Theorem: Let V be a finite-dimensional vector space over a field F.

Let $n = \dim(V)$

So, V has a basis with n vectors in it

Suppose we pick n vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ from V.

① If $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are linearly independent, then they will span V and be a basis for V.

② If $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ span V, then they will be linearly independent and be a basis for V.

Ex: $V = P_2$, $F = \mathbb{R}$

We know (since we showed that $1, x, x^2$ is a basis for P_2) that $\dim(P_2) = 3$.

Consider

$$\left.\begin{aligned} \vec{w}_1 &= 1 \\ \vec{w}_2 &= 1 + x \\ \vec{w}_3 &= 1 + x + x^2 \end{aligned}\right\} \text{3 vectors}$$

Previously, we showed these vectors are linearly independent. Since we have 3 lin. ind. vectors in a 3-dimensional space, they form a basis for P_2.