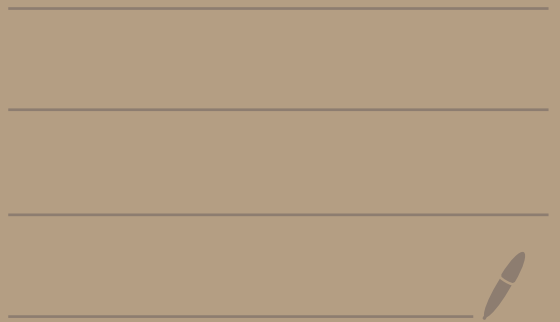


Math 2550-01

4/16/24



T	R
4/16 Eigenvalues	4/18 Test 2 Review
4/23 Test 2	4/25 Eigenvalues
4/30 Topic 9 not on final	5/2 Topic 9 not on final
5/7 Review	5/9 Review
	5/16 Final 12-2

# Topic 8 - Eigenvalues & Eigenvectors

Def: Let  $A$  be an  $n \times n$  matrix.

Suppose that  $\vec{x}$  is in  $\mathbb{R}^n$  and  $\vec{x} \neq \vec{0}$ . If  $A\vec{x} = \lambda\vec{x}$  for some

scalar  $\lambda$ , then we call  $\lambda$  an eigenvalue of  $A$  and  $\vec{x}$  an eigenvector of  $A$  corresponding to  $\lambda$ .

Given an eigenvalue  $\lambda$  of  $A$ , the eigenspace corresponding to  $\lambda$  is

$$E_{\lambda}(A) = \left\{ \vec{x} \mid \vec{x} \in \mathbb{R}^n \text{ and } A\vec{x} = \lambda\vec{x} \right\}$$

$E_{\lambda}(A)$  consists of all eigenvectors corresponding to  $\lambda$  and also the zero vector  $\vec{0}$

$\lambda$   
lambda

Ex: Let  $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$ .

Let  $\vec{x} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ .

Then,

$$A\vec{x} = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$3 \times 3$     $\checkmark$     $3 \times 1$   
answer =  $3 \times 1$

$$= \begin{pmatrix} (0)(-2) + (0)(1) + (-2)(1) \\ (1)(-2) + (2)(1) + (1)(1) \\ (1)(-2) + (0)(1) + (3)(1) \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot \vec{x}$$

So,  $A\vec{x} = 1 \cdot \vec{x}$

$A\vec{x} = \lambda\vec{x}$   
 $\lambda = 1$

So,  $\lambda = 1$  is an eigenvalue of  $A$   
and  $\vec{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$  is an eigenvector  
associated to  $\lambda = 1$ .

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How do we find the eigenvalues  
of an  $n \times n$  matrix  $A$ ?

Suppose  $\lambda$  is an eigenvalue of  $A$

and  $\vec{x} \neq \vec{0}$  and  $A\vec{x} = \lambda\vec{x}$ .

That is,  $\vec{x}$  is an eigenvector of  $A$ .

Then,  $A\vec{x} - \lambda\vec{x} = \vec{0}$ .

So,  $(A - \lambda I_n)\vec{x} = \vec{0}$  where

$I_n$  is the  $n \times n$  identity matrix.

[ Recall  $I_n \vec{x} = \vec{x}$  ]

Thus,

$$(A - \lambda I_n) \vec{x} = \vec{0}$$

$$\vec{x} \neq \vec{0}$$

The only this can happen is if  $A - \lambda I_n$  has no inverse.

Why?

$$\text{Let } B = A - \lambda I_n$$

If  $B^{-1}$  existed then if  $B \vec{x} = \vec{0}$  then  $B^{-1} B \vec{x} = B^{-1} \vec{0}$  and get  $\vec{x} = \vec{0}$ . But  $\vec{x}$  isn't  $\vec{0}$ . Thus,  $B^{-1}$  does not exist.

$$\text{Thus, } \det(A - \lambda I_n) = 0$$

since  $(A - \lambda I_n)^{-1}$  does not exist.

## Summary:

The eigenvalues of  $A$  satisfy the equation  $\det(A - \lambda I_n) = 0$ .

called the characteristic polynomial of  $A$ .

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Ex: Let  $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Let's find the eigenvalues of  $A$ .

We need to solve

$$\det(A - \lambda I_3) = 0$$

↑ because  $A$  is  $3 \times 3$

We have

$$\det(A - \lambda I_3) =$$

$$= \det \left( \underbrace{\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}}_A - \lambda \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3} \right)$$

$$= \det \left( \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$$

expand on  
column 2

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$



$$= -0 + (2-\lambda) \begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} - 0$$

$$\begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$$

$$\begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$$

$$\begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$$

$$= (2-\lambda) \begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda) [(-\lambda)(3-\lambda) - (1)(-2)]$$

$$= (2-\lambda) [\lambda^2 - 3\lambda + 2]$$

let's use this below

$$= 2\lambda^2 - 6\lambda + 4 - \lambda^3 + 3\lambda^2 - 2\lambda$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

characteristic polynomial of A

We want to solve

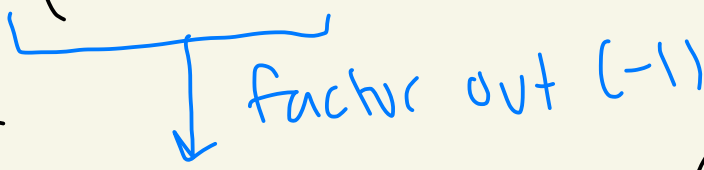
$$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0.$$

From above we have that this factors like this:

$$(2 - \lambda)(\lambda^2 - 3\lambda + 2) = 0$$

which becomes

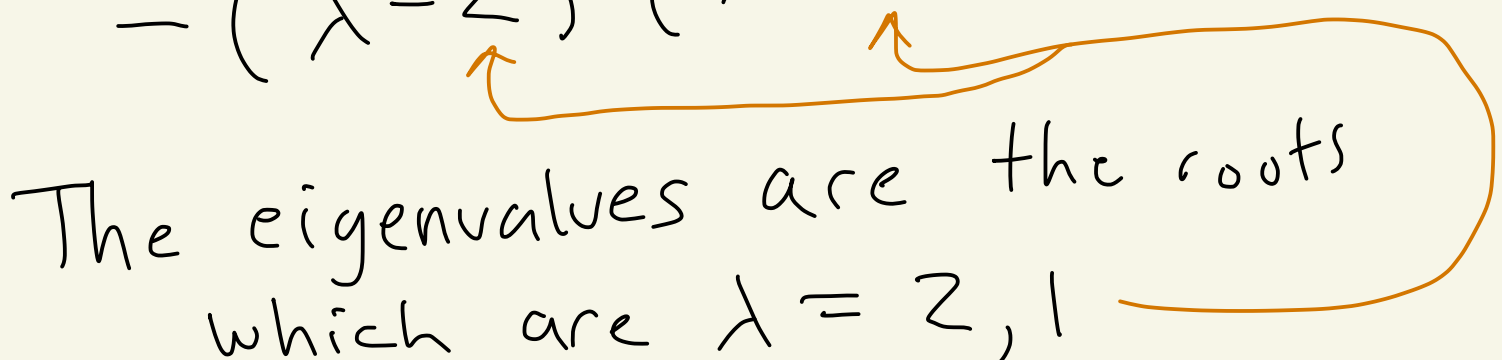
$$(2 - \lambda)(\lambda - 2)(\lambda - 1) = 0$$

or  factor out (-1)

$$-(\lambda - 2)(\lambda - 2)(\lambda - 1) = 0$$

which gives

$$-(\lambda - 2)^2(\lambda - 1) = 0$$

The eigenvalues are the roots which are  $\lambda = 2, 1$  

Let's find eigenvectors  
for the eigenvalues  $\lambda = 2, 1$ .

Let's start with  $\lambda = 1$ ,

We will find a basis for the  
eigenspace  $E_1(A)$ , where

$$E_1(A) = \left\{ \vec{x} \mid \underbrace{A\vec{x} = 1 \cdot \vec{x}}_{\substack{A\vec{x} = \lambda\vec{x} \\ \lambda = 1}} \right\}$$

We have  $A\vec{x} = 1 \cdot \vec{x}$  becomes

$$\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$\underline{\underline{A\vec{x} = 1 \cdot \vec{x}}}$

This gives

$$\begin{pmatrix} 0 \cdot a + 0 \cdot b - 2 \cdot c \\ 1 \cdot a + 2 \cdot b + 1 \cdot c \\ 1 \cdot a + 0 \cdot b + 3 \cdot c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This gives

$$\begin{pmatrix} -2c \\ a + 2b + c \\ a + 3c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This gives

$$\begin{pmatrix} -a - 2c \\ a + b + c \\ a + 2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives

$$\begin{array}{rcl} -a & -2c & = 0 \\ a + b & +c & = 0 \\ a & +2c & = 0 \end{array}$$

Solving we get:

$$\left( \begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array} \right) \xrightarrow{-R_1 \rightarrow R_1} \left( \begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array} \right)$$

$$\begin{array}{l} -R_1 + R_2 \rightarrow R_2 \\ -R_1 + R_3 \rightarrow R_3 \end{array} \rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

This gives

$$a + 2c = 0 \quad (1)$$

$$b - c = 0 \quad (2)$$

$$0 = 0 \quad (3)$$

leading:  $a, b$

free:  $c$

Solving:

$$c = t$$

$$b = c = t$$

$$a = -2c = -2t$$

Thus,  $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$  is in  $E_1(A)$

$$\text{if } \vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

So,  $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$  is a basis for  $E_1(A)$ .

$$\text{And, } \dim(E_1(A)) = 1$$