Math 2550-01
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Topic 8 – Eigenvalues & Eigenvectors

Def: Let A be an $n \times n$ matrix. Suppose that $\vec{x}$ is in $\mathbb{R}^n$ and $\vec{x} \neq \vec{0}$. If $\boxed{A\vec{x} = \lambda x}$ for some scalar λ, then we call λ an eigenvalue of A and $\vec{x}$ an eigenvector of A corresponding to λ.

$\boxed{\lambda \text{ lambda}}$

Given an eigenvalue λ of A, the eigenspace corresponding to λ is

$$E_\lambda(A) = \{ \vec{x} \mid \vec{x} \in \mathbb{R}^n \text{ and } A\vec{x} = \lambda\vec{x} \}$$

[$E_\lambda(A)$ consists of all eigenvectors corresponding to λ and also the zero vector $\vec{0}$]

Ex: Let $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$.

Let $\vec{x} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$.

Then,

$$A\vec{x} = \underbrace{\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}}_{3\times 3} \underbrace{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}_{3\times 1}$$

$3 \times 3 \quad \checkmark \quad 3 \times 1$

answer $= 3 \times 1$

$$= \begin{pmatrix} (0)(-2) + (0)(1) + (-2)(1) \\ (1)(-2) + (2)(1) + (1)(1) \\ (1)(-2) + (0)(1) + (3)(1) \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot \vec{x}$$

So, $A\vec{x} = 1 \cdot \vec{x}$

$\boxed{A\vec{x} = \lambda\vec{x} \quad \lambda = 1}$

So, $\lambda = 1$ is an eigenvalue of A and $\vec{x} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector associated to $\lambda = 1$.

How do we find the eigenvalues of an $n \times n$ matrix A?

Suppose λ is an eigenvalue of A and $\vec{x} \neq \vec{0}$ and $A\vec{x} = \lambda \vec{x}$.

That is, $\vec{x}$ is an eigenvector of A.

Then, $A\vec{x} - \lambda \vec{x} = \vec{0}$.

So, $(A - \lambda I_n)\vec{x} = \vec{0}$ where I_n is the $n \times n$ identity matrix.

[Recall $I_n \vec{x} = \vec{x}$]

Thus,

$$(A - \lambda I_n)\vec{x} = \vec{0} \qquad \vec{x} \neq \vec{0}$$

The only this can happen is if $A - \lambda I_n$ has no inverse.

$\boxed{\text{Why?}}$ Let $B = A - \lambda I_n$

If B^{-1} existed then if $B\vec{x} = \vec{0}$ then $B^{-1}B\vec{x} = B^{-1}\vec{0}$ and get $\vec{x} = \vec{0}$. But $\vec{x}$ isn't $\vec{0}$. Thus, B^{-1} does not exist.

Thus, $\det(A - \lambda I_n) = 0$ since $(A - \lambda I_n)^{-1}$ does not exist.

Summary:

The eigenvalues of A satisfy the equation $\underbrace{\det(A-\lambda I_n)}_{\text{called the characteristic polynomial of } A.} = 0$.

Ex: Let $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Let's find the eigenvalues of A.

We need to solve

$$\det(A - \lambda I_3) = 0$$

because A is 3×3

We have

$$\det(A - \lambda I_3) =$$

$$= \det\left(\underbrace{\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}}_{A} - \lambda \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3}\right)$$

$$= \det\left(\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$$

expand on column 2

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= -\underbrace{0}_{\begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}} + \underbrace{(2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix}}_{\begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}} - \underbrace{0}_{\begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}}$$

$$= (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(-\lambda)(3-\lambda) - (1)(-2)\right]$$

$$= (2-\lambda)\left[\lambda^2 - 3\lambda + 2\right]$$

let's use this below

$$= 2\lambda^2 - 6\lambda + 4 - \lambda^3 + 3\lambda^2 - 2\lambda$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

characteristic polynomial of A

We want to solve

$$-\lambda^3+5\lambda^2-8\lambda+4=0.$$

From above we have that this factors like this:

$$(2-\lambda)(\lambda^2-3\lambda+2)=0$$

which becomes

$$\underbrace{(2-\lambda)}_{\text{factor out } (-1)}(\lambda-2)(\lambda-1)=0$$

or

$$-(\lambda-2)(\lambda-2)(\lambda-1)=0$$

which gives

$$-(\lambda-2)^2(\lambda-1)=0$$

The eigenvalues are the roots which are $\lambda=2,1$

Let's find eigenvectors for the eigenvalues $\lambda = 2, 1$.

Let's start with $\lambda = 1$.

We will find a basis for the eigenspace $E_1(A)$, where

$$E_1(A) = \{ \vec{x} \mid \underbrace{A\vec{x} = 1 \cdot \vec{x}}_{\substack{A\vec{x} = \lambda \vec{x} \\ \lambda = 1}} \}$$

We have $A\vec{x} = 1 \cdot \vec{x}$ becomes

$$\underbrace{\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{A\vec{x}} = \underbrace{1 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{1 \cdot \vec{x}}$$

This gives

$$\begin{pmatrix} 0\cdot a + 0\cdot b - 2\cdot c \\ 1\cdot a + 2\cdot b + 1\cdot c \\ 1\cdot a + 0\cdot b + 3\cdot c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This gives

$$\begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This gives

$$\begin{pmatrix} -a-2c \\ a+b+c \\ a+2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives

$$\boxed{\begin{array}{rrrl} -a & & -2c & = 0 \\ a & +b & +c & = 0 \\ a & & +2c & = 0 \end{array}}$$

Solving we get:

$$\left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \xrightarrow{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow[-R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This gives

$$\boxed{\begin{aligned} a \quad\quad + 2c &= 0 \quad (1) \\ b - c &= 0 \quad (2) \\ 0 &= 0 \quad (3) \end{aligned}}$$

$$\boxed{\begin{array}{l} \text{leading: } a, b \\ \text{free: } c \end{array}}$$

Solving:

$$\boxed{\begin{aligned} & c = t \\ (2) \quad & b = c = t \\ (1) \quad & a = -2c = -2t \end{aligned}}$$

Thus, $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is in $E_1(A)$

if $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

So, $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is a basis for $E_1(A)$.

And, $\dim(E_1(A)) = 1$