Math 2550-01
4/11/24

What's a basis good for?
To make a coordinate system.

<u>Theorem:</u> Let V be a vector space over a field F. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a basis for V. Then given any vector $\vec{v}$ from V there exist unique scalars $c_1, c_2, \ldots, c_n$ from F where

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n$$

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

Previously we showed that

$$\vec{v}_1 = \langle 2, 1 \rangle, \ \vec{v}_2 = \langle -1, 1 \rangle$$

is a basis for $\mathbb{R}^2$.

We also showed that given $\vec{v} = \langle a, b \rangle$ we can write

$$\underbrace{\langle a, b \rangle = \left(\tfrac{1}{3}a + \tfrac{1}{3}b\right)\langle 2, 1 \rangle + \left(-\tfrac{1}{3}a + \tfrac{2}{3}b\right)\langle -1, 1 \rangle}_{\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2}$$

For example,

$$\underbrace{\langle 3, -6 \rangle = (-1) \cdot \langle 2, 1 \rangle + (-5) \cdot \langle -1, 1 \rangle}_{\vec{v} \ = \ c_1 \cdot \vec{v}_1 \ + \ c_2 \cdot \vec{v}_2}$$

If instead your basis was

$$\vec{w}_1 = \langle 1,0 \rangle, \ \vec{w}_2 = \langle 0,1 \rangle \leftarrow \text{standard basis}$$

then

$$\underbrace{\langle 3,-6 \rangle = 3 \cdot \langle 1,0 \rangle + (-6) \cdot \langle 0,1 \rangle}_{\vec{v} = c_1 \cdot \vec{w}_1 + c_2 \cdot \vec{w}_2}$$

Def: Let V be a vector space over a field F. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a basis for V. If we fix this ordering on the basis elements, then we call this an ordered basis for V.

We write

$$\beta = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$$

name of basis

$\beta \leftarrow$ beta

brackets mean that order matters

to denote an ordered basis. Given any vector $\vec{v}$ from V we can write

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

The constants $c_1, c_2, \ldots, c_n$ are called the <u>coordinates of $\vec{v}$ with respect to the basis β.</u>

We write

$$\underbrace{[\vec{v}]_\beta = \langle c_1, c_2, \ldots, c_n \rangle}_{\text{coordinate vector for } \vec{v} \text{ with respect to } \beta.}$$

can also write

$$[\vec{v}]_\beta = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

Let

$$\left.\begin{aligned} \beta &= [\langle 1,0\rangle, \langle 0,1\rangle] \\ \beta' &= [\langle 0,1\rangle, \langle 1,0\rangle] \end{aligned}\right\}$$ two orderings of the standard basis

$$\left.\begin{aligned} \gamma &= [\langle 2,1\rangle, \langle -1,1\rangle] \\ \gamma' &= [\langle -1,1\rangle, \langle 2,1\rangle] \end{aligned}\right\}$$ two orderings of the $\langle 2,1\rangle, \langle -1,1\rangle$ basis

Let $\vec{v} = \langle 3,-6\rangle$.

Then

$$\langle 3,-6\rangle = (3)\cdot\langle 1,0\rangle + (-6)\cdot\langle 0,1\rangle$$

So, $[\langle 3,-6\rangle]_\beta = \langle 3, -6\rangle$

And

$$\langle 3,-6\rangle = (-6)\cdot\langle 0,1\rangle + (3)\cdot\langle 1,0\rangle$$

So,

$$[\langle 3,-6\rangle]_{\beta'} = \langle -6, 3\rangle$$

Also,

$$\langle 3,-6\rangle = (-1)\cdot\langle 2,1\rangle + (-5)\cdot\langle -1,1\rangle$$

So,

$$[\langle 3,-6\rangle]_{\gamma} = \langle -1,-5\rangle$$

$$\boxed{\gamma = [\langle 2,1\rangle, \langle -1,1\rangle]}$$

Also,

$$\langle 3,-6\rangle = (-5)\cdot\langle -1,1\rangle + (-1)\cdot\langle 2,1\rangle$$

Then

$$[\langle 3,-6\rangle]_{\gamma'} = \langle -5,-1\rangle$$

$$\boxed{\gamma' = [\langle -1,1\rangle, \langle 2,1\rangle]}$$

<u>Q</u>: What if you know that $[\vec{v}]_{\gamma} = \langle 1,-1\rangle$,

What is $\vec{v}$?

$$\boxed{\gamma = [\langle 2,1\rangle, \langle -1,1\rangle]}$$

Then

$$\vec{v} = (1)\langle 2,1\rangle + (-1)\cdot\langle -1,1\rangle$$

$$= \langle 3,0\rangle$$

Ex: Let $V = P_2$, $F = \mathbb{R}$.

polys of degree ≤ 2

Let

$\beta = [1, x, x^2]$ standard basis

$\gamma = [1, 1+x, 1+x+x^2]$ another basis we found

Let $\vec{v} = 4 + 2x + 3x^2$.

Find $[\vec{v}]_\beta$ and $[\vec{v}]_\gamma$.

Note

$$\vec{v} = 4 \cdot 1 + 2 \cdot x + 3 \cdot x^2$$

$\beta = [1, x, x^2]$

So,

$$[\vec{v}]_\beta = \langle 4, 2, 3 \rangle$$

To find $[\vec{v}]_\gamma$ we need to solve

$$\underbrace{4+2x+3x^2}_{\vec{v}} = \underbrace{c_1(1)+c_2(1+x)+c_3(1+x+x^2)}_{\gamma=[1,\,1+x,\,1+x+x^2]}$$

This becomes

$$4+2x+3x^2 = c_1+c_2+c_2x+c_3+c_3x+c_3x^2$$

which gives

$$4+2x+3x^2 = (c_1+c_2+c_3)+(c_2+c_3)x+c_3x^2$$

So,

$$\boxed{\begin{aligned} c_1+c_2+c_3&=4 \quad ① \\ c_2+c_3&=2 \quad ② \\ c_3&=3 \quad ③ \end{aligned}} \Rightarrow \boxed{\begin{aligned} c_3&=3 \\ c_2&=2-c_3 \\ &=2-3=-1 \\ c_1&=4-c_2-c_3 \\ &=4+1-3 \\ &=2 \end{aligned}}$$

Thus,

$$4+2x+3x^2 = 2(1)+(-1)(1+x)+3(1+x+x^2)$$

So,

$$[4+2x+3x^2]_\gamma = \langle 2,-1,3\rangle$$

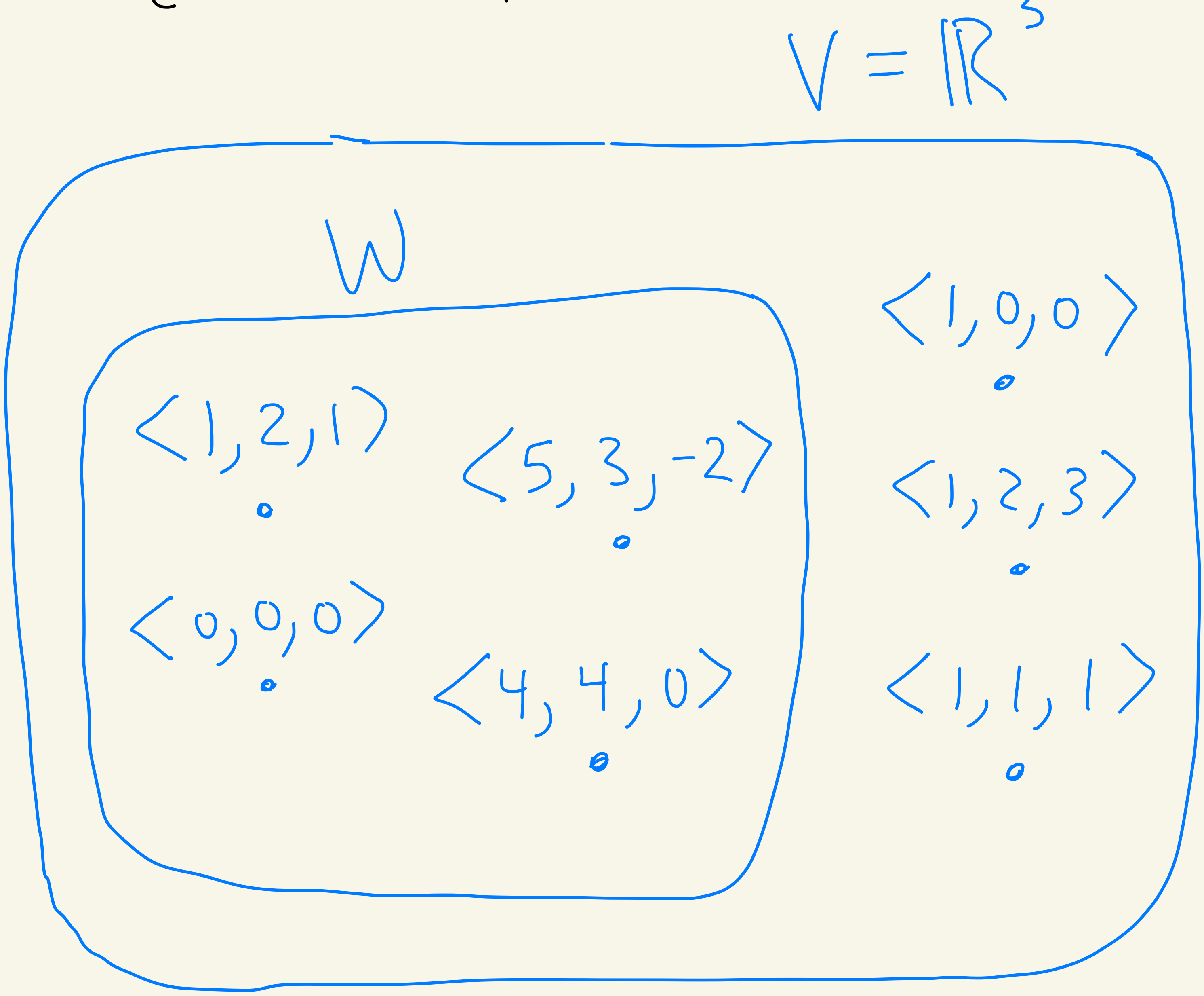
$\gamma = [1, 1+x, 1+x+x^2]$

Let's do an example where we find a basis for a subspace!

$$\boxed{\text{HW 7 – Part 2}}$$

① (b) Let $V = \mathbb{R}^3$, $F = \mathbb{R}$

$$W = \{ \langle a,b,c \rangle \mid b = a + c, \ a,b,c \in \mathbb{R} \}$$



In HW you show W is a subspace of $V = \mathbb{R}^3$. Let's find a basis for W.

Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be in W.

Then, $b = a + c$.

So, $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ a+c \\ c \end{pmatrix} = \begin{pmatrix} a \\ a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c \\ c \end{pmatrix}$

$$= a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Thus, the vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ span all of W.

Are $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ linearly independent?

We need to solve

$$c_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

for c_1, c_2.

This becomes

$$\begin{pmatrix} c_1 \\ c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives

$$\begin{pmatrix} c_1 \\ c_1 + c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \leftarrow c_1 = 0 \\ \leftarrow c_1 + c_2 = 0 \\ \leftarrow c_2 = 0 \end{array}$$

So, $c_1 = 0, c_2 = 0$.

Thus, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are linearly independent.

Thus, a basis for W is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Therefore, $\dim(W) = 2$.

W is a 2-dimensional space

inside a 3-dimensional space

$V = \mathbb{R}^3.$