Math 2550-01
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3/5 Finish determinants	3/7 Review
3/12 Test 1	3/14 topic 6 starts

Last time we showed that

$$\det\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} = -1$$

by expanding on column 2.

Let's compute this again but expand on a row.

Ex:

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} =$$

$$= (3)\cdot\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (1)\cdot\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\cdot\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \qquad \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \qquad \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= 3\cdot\left[(-4)(-2) - (3)(4)\right] - 1\cdot\left[(-2)(-2) - (3)(5)\right]$$

$$+ 0$$

$$= 3\left[8-12\right] - \left[4-15\right]$$

$$= 3\left[-4\right] - \left[-11\right]$$

$$= -12+11 = -1$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

Ex:

$$\det\begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

expand on row 1

$$= 1 \cdot \begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & -2 \\ 1 & 2 & 0 \end{vmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$$-0\cdot\begin{vmatrix}2&3&0\\1&0&-2\\1&2&0\end{vmatrix} \leftarrow \begin{pmatrix}1&0&0&-1\\2&1&3&0\\1&0&0&-2\\1&1&2&0\end{pmatrix}$$

$$+0\cdot\begin{vmatrix}2&1&0\\1&0&-2\\1&1&0\end{vmatrix} \leftarrow \begin{pmatrix}1&0&0&-1\\2&1&3&0\\1&0&0&-2\\1&1&2&0\end{pmatrix}$$

$$-(-1)\cdot\begin{vmatrix}2&1&3\\1&0&0\\1&1&2\end{vmatrix} \leftarrow \begin{pmatrix}1&0&0&-1\\2&1&3&0\\1&0&0&-2\\1&1&2&0\end{pmatrix}$$

$$=\begin{vmatrix}1&3&0\\0&0&-2\\1&2&0\end{vmatrix}+\begin{vmatrix}2&1&3\\1&0&0\\1&1&2\end{vmatrix}$$

$\boxed{\text{expand on row 2 on both}}$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= -0\cdot\underbrace{\begin{vmatrix}3&0\\2&0\end{vmatrix}}_{\begin{pmatrix}1&3&0\\0&0&-2\\1&2&0\end{pmatrix}} + 0\cdot\underbrace{\begin{vmatrix}1&0\\1&0\end{vmatrix}}_{\begin{pmatrix}1&3&0\\0&0&-2\\1&2&0\end{pmatrix}} - (-2)\cdot\underbrace{\begin{vmatrix}1&3\\1&2\end{vmatrix}}_{\begin{pmatrix}1&3&0\\0&0&-2\\1&2&0\end{pmatrix}}$$

$$-1\cdot\underbrace{\begin{vmatrix}1&3\\1&2\end{vmatrix}}_{\begin{pmatrix}2&1&3\\1&0&0\\1&1&2\end{pmatrix}} + 0\cdot\underbrace{\begin{vmatrix}2&3\\1&2\end{vmatrix}}_{\begin{pmatrix}2&1&3\\1&0&0\\1&1&2\end{pmatrix}} - 0\cdot\underbrace{\begin{vmatrix}2&1\\1&1\end{vmatrix}}_{\begin{pmatrix}2&1&3\\1&0&0\\1&1&2\end{pmatrix}}$$

$$= 2\cdot\begin{vmatrix}1&3\\1&2\end{vmatrix} - \begin{vmatrix}1&3\\1&2\end{vmatrix}$$

$$= 2\cdot\left[(1)(2)-(3)(1)\right] - \left[(1)(2)-(3)(1)\right]$$

$= 2 \cdot [2-3] - [2-3]$

$= 2[-1] - [-1] = -2+1 = -1$

Properties of determinants

Let A and B be $n \times n$ matrices.

① $\det(A^T) = \det(A)$

② $\det(AB) = \det(A) \cdot \det(B)$

③ If A^{-1} exists, then $\det(A) \neq 0$

④ If $\det(A) \neq 0$, then A^{-1} exists

[So, if $\det(A) = 0$, then A^{-1} does not exist]

⑤ If A^{-1} exists, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$

⑥ $\det(I_n) = 1$ ← I_n is $n \times n$ identity matrix

⑦ If A has a row or column of zeros, then $\det(A) = 0$

ex:

$$\det\begin{pmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} = 0$$

column of zeros

⑧ If a row is a multiple of another row, or a column is a multiple of another column in A, then $\det(A) = 0$

ex:

$$\det\begin{pmatrix} 1 & 3 & 5 \\ 0 & 0 & 2 \\ -1 & -3 & 7 \end{pmatrix} = 0$$

(column 2) = 3·(column 1)

Formula for A^{-1} when A is 2×2

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then, $\det(A) = ad - bc$

If $\det(A) \neq 0$, then A^{-1} will exist and

$$A^{-1} = \underbrace{\frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{\frac{1}{\det(A)} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Ex: $A = \begin{pmatrix} 1 & 5 \\ 6 & -1 \end{pmatrix}$

$$\det(A) = (1)(-1) - (5)(6) = \underbrace{-31}_{\substack{\text{not } 0 \\ \text{so } A^{-1} \\ \text{exists}}}$$

$$A^{-1} = \frac{1}{-31} \cdot \begin{pmatrix} -1 & -5 \\ -6 & 1 \end{pmatrix} = \begin{pmatrix} 1/31 & 5/31 \\ 6/31 & -1/31 \end{pmatrix}$$

In general, if A is $n \times n$ and $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \cdot M$$

where M is called the adjugate matrix