

Math 2550-01

3/28/24

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We are going to define a basis which is a way to create a coordinate system in a vector space

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Def: Let  $V$  be a vector space over a field  $F$ .

Let  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  be vectors in  $V$ . We call  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  a basis for  $V$  if two conditions hold:

①  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  span  $V$

this means every vector  $\vec{v}$  in  $V$  can be expressed in the form

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n$$

(2)  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  are linearly independent.

none of the  $\vec{v}_i$  can be expressed as linear combos of the other vectors, ie no redundancies

Ex: Let  $V = \mathbb{R}^2$ ,  $F = \mathbb{R}$ .

Let  $\vec{v}_1 = \langle 1, 0 \rangle$ ,  $\vec{v}_2 = \langle 0, 1 \rangle$ .

(1) we showed  $\vec{v}_1, \vec{v}_2$  span  $V = \mathbb{R}^2$

(2) we showed  $\vec{v}_1, \vec{v}_2$  are lin. ind.

So,  $\vec{v}_1 = \langle 1, 0 \rangle$ ,  $\vec{v}_2 = \langle 0, 1 \rangle$  is a basis for  $V = \mathbb{R}^2$ .

Any vector  $\vec{v} = \langle a, b \rangle$  then

$$\vec{v} = \langle a, b \rangle = a \langle 1, 0 \rangle + b \langle 0, 1 \rangle$$

$$= a\vec{v}_1 + b\vec{v}_2$$

Ex: Let  $V = \mathbb{R}^2$ ,  $F = \mathbb{R}$

Let  $\vec{v}_1 = \langle 2, 1 \rangle$ ,  $\vec{v}_2 = \langle -1, 1 \rangle$ .

① We showed previously that  $\vec{v}_1, \vec{v}_2$  span  $V = \mathbb{R}^2$ . In fact, we showed that any vector  $\langle a, b \rangle$  can be written like this:

$$\begin{aligned}\langle a, b \rangle &= \left(\frac{1}{3}a + \frac{1}{3}b\right)\langle 2, 1 \rangle + \left(-\frac{1}{3}a + \frac{2}{3}b\right)\langle -1, 1 \rangle \\ &= \left(\frac{1}{3}a + \frac{1}{3}b\right)\vec{v}_1 + \left(-\frac{1}{3}a + \frac{2}{3}b\right)\vec{v}_2\end{aligned}$$

② We never showed these vectors are linearly independent.

We need to solve

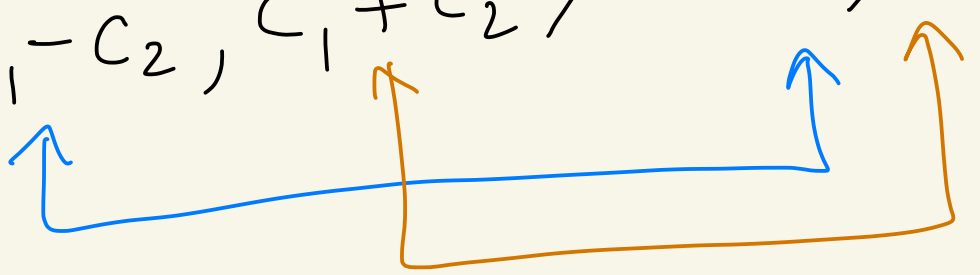
$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$$

for  $c_1, c_2$ .

This becomes

$$c_1 \langle 2, 1 \rangle + c_2 \langle -1, 1 \rangle = \langle 0, 0 \rangle$$

which is

$$\langle 2c_1 - c_2, c_1 + c_2 \rangle = \langle 0, 0 \rangle$$
A diagram with three arrows pointing from the components of the vector equation to the boxed system. A blue arrow points from the first component  $2c_1 - c_2$  to the first equation  $2c_1 - c_2 = 0$ . An orange arrow points from the second component  $c_1 + c_2$  to the second equation  $c_1 + c_2 = 0$ . A blue arrow points from the zero component of the second component  $0$  to the zero on the right side of the second equation.

This gives

$$\begin{cases} 2c_1 - c_2 = 0 \\ c_1 + c_2 = 0 \end{cases}$$

Solving:

$$\left( \begin{array}{cc|c} 2 & -1 & 0 \\ 1 & 1 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 2 & -1 & 0 \end{array} \right)$$

$$\xrightarrow{-2R_1 + R_2 \rightarrow R_2} \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -3 & 0 \end{array} \right)$$

$$\xrightarrow{-\frac{1}{3}R_2 \rightarrow R_2} \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right)$$

This gives

$$\boxed{\begin{array}{l} c_1 + c_2 = 0 \\ c_2 = 0 \end{array}} \begin{array}{l} \textcircled{1} \\ \textcircled{2} \end{array}$$

leading:  $c_1, c_2$   
no free var.

So,

$$\textcircled{2} \quad c_2 = 0$$

$$\textcircled{1} \quad c_1 = -c_2 = -(0) = 0.$$

So, the only solution to

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$$

is  $c_1 = 0, c_2 = 0$ .

Thus,  $\vec{v}_1 = \langle 2, 1 \rangle, \vec{v}_2 = \langle -1, 1 \rangle$   
are linearly independent.

Thus, from above  $\vec{v}_1 = \langle 2, 1 \rangle,$   
 $\vec{v}_2 = \langle -1, 1 \rangle$  are a basis

for  $V = \mathbb{R}^2$

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Theorem: Let  $V$  be a vector space over a field  $F$ . Let  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  be a basis with  $n$  vectors. Then any other basis for  $V$  will also have exactly  $n$  vectors in it.

All of the bases for  $V$  have the same # of vectors

Ex:  $V = \mathbb{R}^2, F = \mathbb{R}$

basis #1:  $\vec{v}_1 = \langle 1, 0 \rangle, \vec{v}_2 = \langle 0, 1 \rangle$

basis #2:  $\vec{v}_1 = \langle 2, 1 \rangle, \vec{v}_2 = \langle -1, 1 \rangle$

Here both bases have  $n=2$  vectors  
All bases for  $V = \mathbb{R}^2$  have 2 vectors in them.



Def: Let  $V$  be a vector space over a field  $F$ . If there exists a basis  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  with  $n$  vectors for  $V$ , then we call  $V$  a finite-dimensional vector space and we say that  $V$  has dimension  $n$  and write  $\dim(V) = n$ .

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Ex:  $V = \mathbb{R}^2, F = \mathbb{R}$

Then,  $\dim(\mathbb{R}^2) = 2$  since  $\vec{v}_1 = \langle 1, 0 \rangle, \vec{v}_2 = \langle 0, 1 \rangle$  is a basis for  $V = \mathbb{R}^2$  and it

has 2 vectors in it

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Ex: Let

$V = P_2$  ← all polynomials  
of degree  $\leq 2$

$F = \mathbb{R}$

Let  $\vec{v}_1 = 1, \vec{v}_2 = x, \vec{v}_3 = x^2$ .

Claim:  $\vec{v}_1 = 1, \vec{v}_2 = x, \vec{v}_3 = x^2$  is  
a basis for  $V = P_2$

proof:

① (spanning)

Given any vector  $\vec{v} = a + bx + cx^2$   
in  $P_2$  we have that

$$\begin{aligned}\vec{v} &= a \cdot 1 + b \cdot x + c \cdot x^2 \\ &= a \cdot \vec{v}_1 + b \cdot \vec{v}_2 + c \cdot \vec{v}_3\end{aligned}$$

So,  $\vec{v}_1 = 1, \vec{v}_2 = x, \vec{v}_3 = x^2$  span  $V = P_2$ .

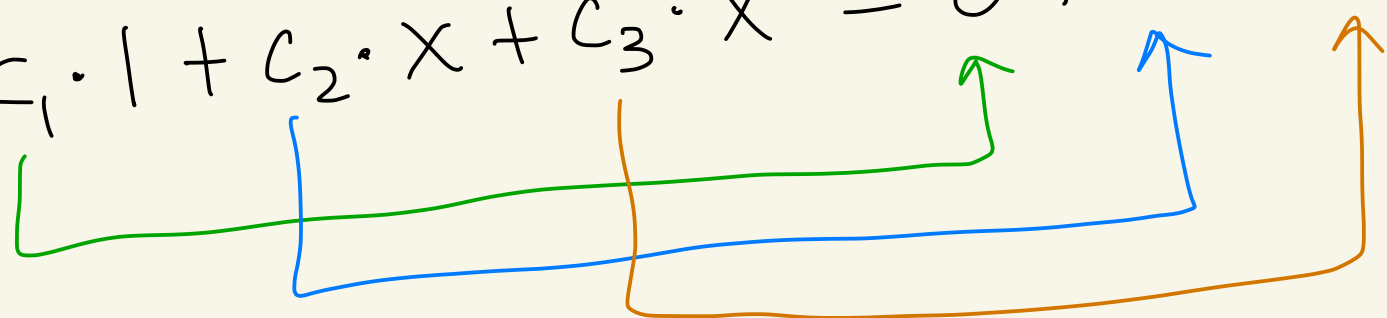
② (linear independence)

Let's solve

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

for  $c_1, c_2, c_3$ .

This becomes

$$c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 = 0 + 0x + 0x^2$$


This can only happen when  
 $c_1 = 0, c_2 = 0, c_3 = 0$ .

Thus,  $\vec{v}_1 = 1, \vec{v}_2 = x, \vec{v}_3 = x^2$   
are linearly independent.

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Claim —

Note:

Since  $V = P_2$  has basis

$\vec{v}_1 = 1, \vec{v}_2 = x, \vec{v}_3 = x^2$  we

know  $\dim(P_2) = 3.$