Math 2550-01
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We are going to define a basis which is a way to create a coordinate system in a vector space

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<u>Def:</u> Let V be a vector space over a field F. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be vectors in V. We call $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ a basis for V if two conditions hold:

(1) $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span V

[this means every vector $\vec{v}$ in V can be expressed in the form
$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_n \vec{v}_n$$
]

(2) $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent.

[none of the $\vec{v}_i$ can be expressed as linear combos of the other vectors, ie no redundencies]

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

Let $\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$.

(1) we showed $\vec{v}_1, \vec{v}_2$ span $V = \mathbb{R}^2$

(2) we showed $\vec{v}_1, \vec{v}_2$ are lin. ind.

So, $\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$ is a basis for $V = \mathbb{R}^2$.

Any vector $\vec{v} = \langle a, b \rangle$ then

$$\vec{v} = \langle a, b \rangle = a \langle 1, 0 \rangle + b \langle 0, 1 \rangle$$

$= a\vec{v}_1 + b\vec{v}_2$

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$

Let $\vec{v}_1 = \langle 2,1 \rangle$, $\vec{v}_2 = \langle -1,1 \rangle$.

① We showed previously that $\vec{v}_1, \vec{v}_2$ span $V = \mathbb{R}^2$. In fact, we showed that any vector $\langle a,b \rangle$ can be written like this:

$$\langle a,b \rangle = \left(\tfrac{1}{3}a + \tfrac{1}{3}b\right)\langle 2,1 \rangle + \left(-\tfrac{1}{3}a + \tfrac{2}{3}b\right)\langle -1,1 \rangle$$
$$= \left(\tfrac{1}{3}a + \tfrac{1}{3}b\right)\vec{v}_1 + \left(-\tfrac{1}{3}a + \tfrac{2}{3}b\right)\vec{v}_2$$

② We never showed these vectors are linearly independent.

We need to solve

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$$

for c_1, c_2.

This becomes

$$c_1 \langle 2, 1 \rangle + c_2 \langle -1, 1 \rangle = \langle 0, 0 \rangle$$

which is

$$\langle 2c_1 - c_2, c_1 + c_2 \rangle = \langle 0, 0 \rangle$$

This gives

$$\boxed{\begin{aligned} 2c_1 - c_2 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}}$$

Solving:

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -3 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 &= 0 \quad (1) \\ c_2 &= 0 \quad (2) \end{aligned}}$$

leading: c_1, c_2
no free var.

So,

(2) $c_2 = 0$

(1) $c_1 = -c_2 = -(0) = 0$.

So, the only solution to
$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$

is $c_1 = 0$, $c_2 = 0$.

Thus, $\vec{v}_1 = \langle 2, 1 \rangle$, $\vec{v}_2 = \langle -1, 1 \rangle$ are linearly independent.

Thus, from above $\vec{v}_1 = \langle 2, 1 \rangle$, $\vec{v}_2 = \langle -1, 1 \rangle$ are a basis for $V = \mathbb{R}^2$

Theorem: Let V be a vector space over a field F. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a basis with n vectors. Then any other basis for V will also have exactly n vectors in it.

All of the bases for V have the same # of vectors

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

basis #1: $\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$

basis #2: $\vec{v}_1 = \langle 2, 1 \rangle$, $\vec{v}_2 = \langle -1, 1 \rangle$

Here both bases have $n = 2$ vectors

All bases for $V = \mathbb{R}^2$ have 2 vectors in them.

Def: Let V be a vector space over a field F. If there exists a basis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ with n vectors for V, then we call V a finite-dimensional vector space and we say that V has dimension n and write $\dim(V) = n$.

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

Then, $\dim(\mathbb{R}^2) = 2$ since $\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$ is a basis for $V = \mathbb{R}^2$ and it has 2 vectors in it

Ex: Let

$V = P_2$ ← all polynomials of degree ≤ 2

$F = \mathbb{R}$

Let $\vec{v}_1 = 1$, $\vec{v}_2 = x$, $\vec{v}_3 = x^2$.

$\boxed{\text{Claim: } \vec{v}_1 = 1, \vec{v}_2 = x, \vec{v}_3 = x^2 \text{ is a basis for } V = P_2}$

proof:

① (spanning)

Given any vector $\vec{v} = a + bx + cx^2$ in P_2 we have that

$$\vec{v} = a\cdot 1 + b\cdot x + c\cdot x^2$$
$$= a\cdot\vec{v}_1 + b\cdot\vec{v}_2 + c\cdot\vec{v}_3$$

So, $\vec{v}_1 = 1$, $\vec{v}_2 = x$, $\vec{v}_3 = x^2$ span $V = P_2$.

② (linear independence)

Let's solve
$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$
for c_1, c_2, c_3.

This becomes
$$c_1\cdot 1 + c_2\cdot x + c_3\cdot x^2 = 0 + 0x + 0x^2$$

This can only happen when
$$c_1 = 0,\ c_2 = 0,\ c_3 = 0.$$

Thus, $\vec{v}_1 = 1$, $\vec{v}_2 = x$, $\vec{v}_3 = x^2$ are linearly independent.

$\boxed{\text{Claim}}$

Note:

Since $V = P_2$ has basis $\vec{v}_1 = 1$, $\vec{v}_2 = x$, $\vec{v}_3 = x^2$ we know $\dim(P_2) = 3$.