Math 2550-01
3/26/24

<u>Def:</u> Let V be a vector space over a field F.

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be vectors in V.

We say that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are <u>linearly dependent</u> if there exist scalars/numbers $c_1, c_2, \ldots, c_n$ from F, that are not all equal to zero (but some can be zero) where

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$$

If no such scalars exist then the vectors are called <u>linearly independent</u>.

Reformulated:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$$

always has at least one solution, it is $c_1 = 0, c_2 = 0, \dots, c_n = 0$

If there are more solutions, then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly dependent.

If that's the only solution, then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent.

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

Let $\vec{v}_1 = \langle 3, -5 \rangle$, $\vec{v}_2 = \langle -6, 10 \rangle$

Some solutions to

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$$

are:

$$\underbrace{0 \cdot \langle 3, -5 \rangle + 0 \cdot \langle -6, 10 \rangle = \langle 0, 0 \rangle}_{0\vec{v}_1 + 0\vec{v}_2 = \vec{0}}$$

$$\underbrace{2 \cdot \langle 3, -5 \rangle + 1 \cdot \langle -6, 10 \rangle = \langle 0, 0 \rangle}_{2 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 = \vec{0}}$$

Since $2 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 = \vec{0}$ the vectors $\vec{v}_1 = \langle 3, -5 \rangle$, $\vec{v}_2 = \langle -6, 10 \rangle$

are linearly dependent.

Note: From $2\cdot\vec{v}_1 + 1\cdot\vec{v}_2 = \vec{0}$

gives $\vec{v}_1 = -\frac{1}{2}\vec{v}_2$ ← $\vec{v}_1$ is a linear combo of $\vec{v}_2$

or $\vec{v}_2 = -2\vec{v}_1$ ← $\vec{v}_2$ is a linear combo of $\vec{v}_1$

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

Let $\vec{v}_1 = \langle 1,-1\rangle$, $\vec{v}_2 = \langle 0,1\rangle$, $\vec{v}_3 = \langle 2,-1\rangle$

Then,

$$\underbrace{2\cdot\langle 1,-1\rangle + 1\cdot\langle 0,1\rangle - 1\cdot\langle 2,-1\rangle = \langle 0,0\rangle}$$

$$2\vec{v}_1 + 1\cdot\vec{v}_2 - 1\cdot\vec{v}_3 = \vec{0}$$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

where $c_1 = 2$, $c_2 = 1$, $c_3 = -1$

are not all equal to zero.

So, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent

<u>Note:</u> $\vec{v}_1 = -\frac{1}{2}\vec{v}_2 + \frac{1}{2}\vec{v}_3$

$\vec{v}_1$ is a linear combo of $\vec{v}_2$ & $\vec{v}_3$

$$\left.\begin{aligned} \vec{v}_2 &= -2\vec{v}_1 + 1\cdot\vec{v}_3 \\ \vec{v}_3 &= 2\vec{v}_1 + 1\cdot\vec{v}_2 \end{aligned}\right\} \text{other linear combos.}$$

<u>Ex:</u> $V = \mathbb{R}^3$, $F = \mathbb{R}$

Let $\vec{v}_1 = \langle 1, 1, 1 \rangle$, $\vec{v}_2 = \langle 1, 0, 1 \rangle$, $\vec{v}_3 = \langle 1, \frac{4}{3}, 1 \rangle$. Are these vectors lin. dep. or lin. indep.?

We want to find the solutions to

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

which is

$$c_1\langle 1,1,1\rangle + c_2\langle 1,0,1\rangle + c_3\langle 1,\tfrac{4}{3},1\rangle = \langle 0,0,0\rangle$$

which becomes

$$\langle c_1,c_1,c_1\rangle + \langle c_2,0,c_2\rangle + \langle c_3,\tfrac{4}{3}c_3,c_3\rangle = \langle 0,0,0\rangle$$

which becomes

$$\langle c_1+c_2+c_3,\ c_1+\tfrac{4}{3}c_3,\ c_1+c_2+c_3\rangle = \langle 0,0,0\rangle$$

This gives

$$\boxed{\begin{aligned} c_1+c_2+c_3 &= 0\\ c_1 \qquad\ + \tfrac{4}{3}c_3 &= 0\\ c_1+c_2+c_3 &= 0 \end{aligned}}$$

Solving:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0\\ 1 & 0 & 4/3 & 0\\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow[-R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0\\ 0 & -1 & 1/3 & 0\\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Continuing we get:

$$c_1 + c_2 + c_3 = 0 \quad (1)$$
$$c_2 - \frac{1}{3} c_3 = 0 \quad (2)$$
$$0 = 0 \quad (3)$$

leading variables
c_1, c_2
free variable
c_3

$$c_1 = -c_2 - c_3 \quad (1)$$
$$c_2 = \frac{1}{3} c_3 \quad (2)$$
$$c_3 = t \quad (3)$$

Solving:

(3) $c_3 = t$

(2) $c_2 = \frac{1}{3} c_3 = \frac{1}{3} t$

(1) $c_1 = -c_2 - c_3 = -\frac{1}{3} t - t = -\frac{4}{3} t$

Plugging this back into

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

gives

$$\left(-\frac{4}{3}t\right)\vec{v}_1 + \left(\frac{1}{3}t\right)\vec{v}_2 + (t)\vec{v}_3 = \vec{0}$$

for any real number t.

For example, when $t=3$ we get

$$\boxed{-4\vec{v}_1 + 1\cdot\vec{v}_2 + 3\cdot\vec{v}_3 = \vec{0}}$$

So, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

<u>Ex:</u> $V = \mathbb{R}^2$, $F = \mathbb{R}$

Let $\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$.

Are these vectors lin. dep. or lin. indep.?

We need to solve

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$$

which is

$$c_1 \langle 1, 0 \rangle + c_2 \langle 0, 1 \rangle = \langle 0, 0 \rangle$$

This becomes

$$\langle c_1, 0 \rangle + \langle 0, c_2 \rangle = \langle 0, 0 \rangle$$

This becomes

$$\langle c_1, c_2 \rangle = \langle 0, 0 \rangle$$

So, $c_1 = 0$, $c_2 = 0$.

Thus the only solution to

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$$

is $c_1 = 0, c_2 = 0$.

only eqn we can make is $0\vec{v}_1 + 0\vec{v}_2 = \vec{0}$

So, $\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$ are linearly independent.

<u>Ex</u>: $V = P_2$ ← polynomials up to degree 2

$F = \mathbb{R}$

Let $\vec{v}_1 = 1$, $\vec{v}_2 = 1 + x$, $\vec{v}_3 = 1 + x + x^2$

Are these vectors linearly dependent or linearly independent?

We must solve

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

Which is

$$c_1(1) + c_2(1+x) + c_3(1+x+x^2) = \underbrace{0+0x+0x^2}_{\vec{0}}$$

which gives

$$c_1 + c_2 + c_2 x + c_3 + c_3 x + c_3 x^2 = 0 + 0x + 0x^2$$

which gives

$$(c_1+c_2+c_3) + (c_2+c_3)x + c_3 x^2 = 0 + 0x + 0x^2$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 0 \quad (1)\\ c_2 + c_3 &= 0 \quad (2)\\ c_3 &= 0 \quad (3)\end{aligned}}$$

reduced with no free variables

Solving gives

(3) $c_3 = 0$

(2) $c_2 = -c_3 = -(0) = 0$

(1) $c_1 = -c_2 - c_3 = -0 - 0 = 0$

Thus, the only solution to

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

is $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

So, $\vec{v}_1 = 1$, $\vec{v}_2 = 1+x$, $\vec{v}_3 = 1+x+x^2$

are linearly independent.

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

Then, $\vec{v}_1 = \langle 1,1 \rangle$, $\vec{v}_2 = \langle 6,7 \rangle$

$\vec{v}_3 = \langle -2,-2 \rangle$

are linearly dependent because

$$2 \cdot \langle 1,1 \rangle + 0 \cdot \langle 6,7 \rangle + 1 \cdot \langle -2,-2 \rangle = \langle 0,0 \rangle$$

That is,

$$2 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 1 \cdot \vec{v}_3 = \vec{0}$$

$c_1 = 2, c_2 = 0, c_3 = 1$

not all zero
