Math 2550-01

3/21/24

Topic 7 - Spanning, linear independence, basis

We are going to develop what a coordinate system is. It will be called a basis.

<u>Def:</u> Let V be a vector space over a field F. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be vectors from V.

① A vector $\vec{v}$ is said to be in the <u>span</u> of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if we can write

$$\vec{v} = \underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n v_n}_{\text{is called a linear combination of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n}$$

where $c_1, c_2, \ldots, c_n$ are scalars from F.

(2) The <u>span</u> of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is the set

$$\operatorname{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}) =$$

$$= \left\{ c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \;\middle|\; \begin{array}{l} c_1, c_2, \ldots, c_n \text{ are} \\ \text{scalars from } F \end{array} \right\}$$

(3) If $W = \operatorname{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\})$ then we say that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span W.

Ex: Let $V=\mathbb{R}^2$, $F=\mathbb{R}$.

Let $\vec{v}_1 = \langle 1, 0 \rangle$.

Q: Is $\vec{w} = \langle -5, 0 \rangle$ in the span of $\vec{v}_1$?

Can we write

$$\underbrace{\langle -5, 0 \rangle}_{\vec{w}} = \underbrace{c_1 \langle 1, 0 \rangle}_{c_1 \vec{v}_1}$$

for some scalar c_1?

Yes, $\underbrace{\langle -5, 0 \rangle = (-5) \langle 1, 0 \rangle}_{\vec{w} = -5\vec{v}_1.}$

So, $\vec{w}$ is in the span of $\vec{v}_1$.

Q: Is $\vec{z} = \langle 1, -1 \rangle$ in the span of $\vec{v}_1$?

We would need to solve

$$\underbrace{\langle 1, -1 \rangle = c_1 \langle 1, 0 \rangle}_{\vec{z} = c_1 \vec{v}_1}$$

This would become $\langle 1, -1 \rangle = \langle c_1, 0 \rangle$

can't happen

So, no, $\vec{z}$ is not in the span of $\vec{v}_1$.

Q: What is the span of $\vec{v}_1$?

$$\text{span}(\{\vec{v}_1\}) = \{ c_1 \vec{v}_1 \mid c_1 \in \mathbb{R} \}$$
$$= \{ c_1 \langle 1, 0 \rangle \mid c_1 \in \mathbb{R} \}$$

$$= \{ \langle c_1, 0 \rangle \mid c_1 \in \mathbb{R} \}$$

The span of $\vec{v}_1$ consists of all vectors of the form $\langle c_1, 0 \rangle$ where c_1 is a scalar in $\mathbb{R}$.

<u>Ex:</u> Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

Let $\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$.

<u>Q:</u> Is $\vec{w} = \langle 10, -3 \rangle$ in the span of $\vec{v}_1, \vec{v}_2$?

Can we write

$$\underbrace{\langle 10, -3 \rangle = c_1 \langle 1, 0 \rangle + c_2 \langle 0, 1 \rangle}_{\vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2}$$

This would become

$$\langle 10, -3 \rangle = \langle c_1, 0 \rangle + \langle 0, c_2 \rangle$$

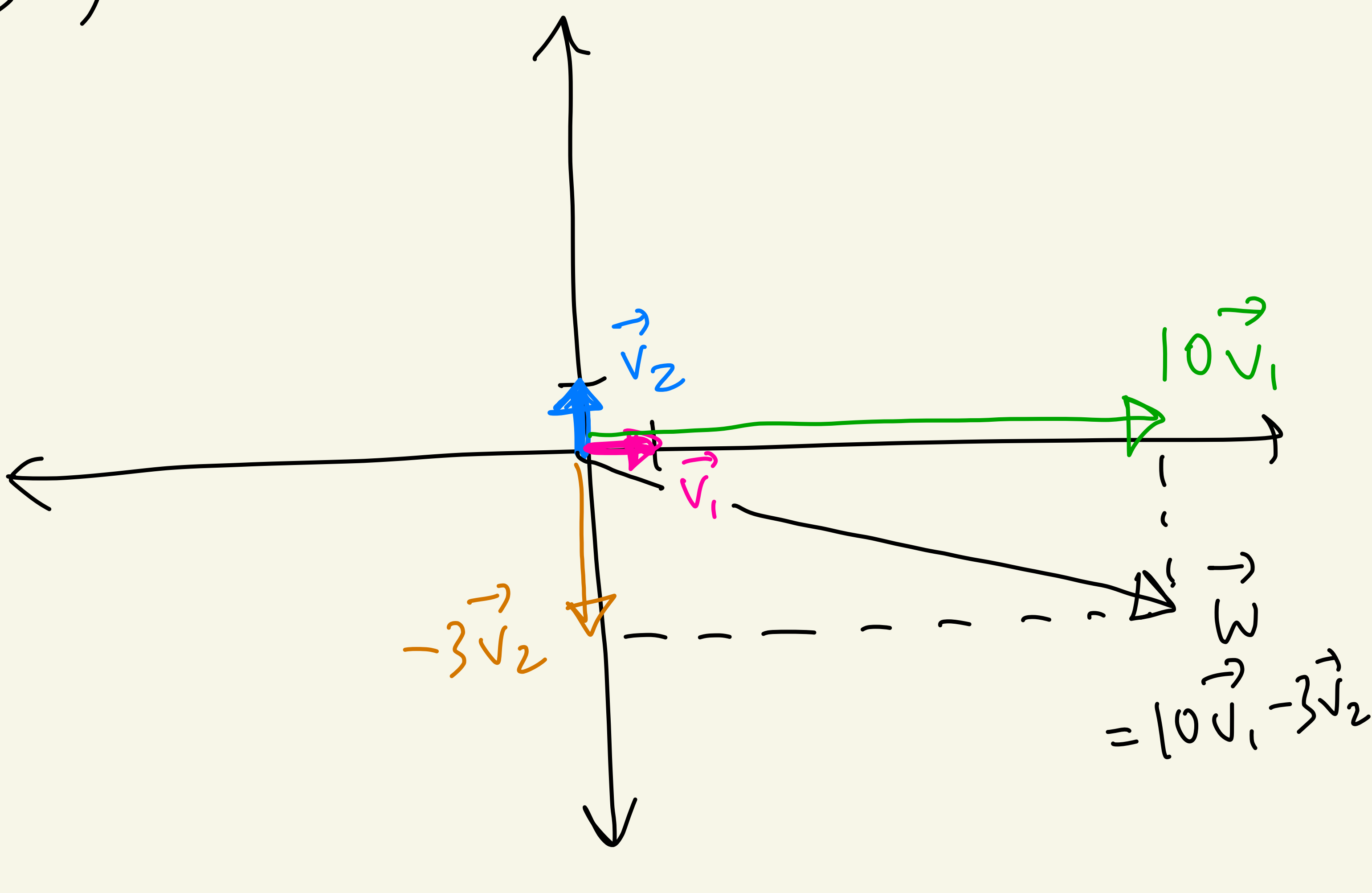
which gives

$$\langle 10, -3 \rangle = \langle c_1, c_2 \rangle$$

So, $c_1 = 10$, $c_2 = -3$.

That is, $\vec{w} = 10\vec{v}_1 - 3\vec{v}_2$

So, $\vec{w}$ is in the span of $\vec{v}_1, \vec{v}_2$



Q: What is the span of $\vec{v}_1, \vec{v}_2$?

$$\text{span}(\{\vec{v}_1, \vec{v}_2\}) = \{c_1\vec{v}_1 + c_2\vec{v}_2 \mid c_1, c_2 \in \mathbb{R}\}$$

$$= \{c_1\langle 1,0\rangle + c_2\langle 0,1\rangle \mid c_1, c_2 \in \mathbb{R}\}$$

$$= \{\langle c_1, 0\rangle + \langle 0, c_2\rangle \mid c_1, c_2 \in \mathbb{R}\}$$

$$= \{\langle c_1, c_2\rangle \mid c_1, c_2 \in \mathbb{R}\}$$

$$= \mathbb{R}^2$$

Another way:

Given any vector $\langle a,b\rangle$ from $\mathbb{R}^2$ we get

$$\langle a,b\rangle = \underbrace{a\langle 1,0\rangle + b\langle 0,1\rangle}_{a\vec{v}_1 + b\vec{v}_2}$$

So any vectors in $\mathbb{R}^2$ is in the span of $\vec{v}_1, \vec{v}_2$.

For example,

$$\langle 2, 5 \rangle = 2\vec{v}_1 + 5\vec{v}_2$$

Summary: $\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$ span all of $V = \mathbb{R}^2$.

<u>Ex:</u> Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

Let $\vec{v}_1 = \langle 2, 1 \rangle$, $\vec{v}_2 = \langle -1, 1 \rangle$.

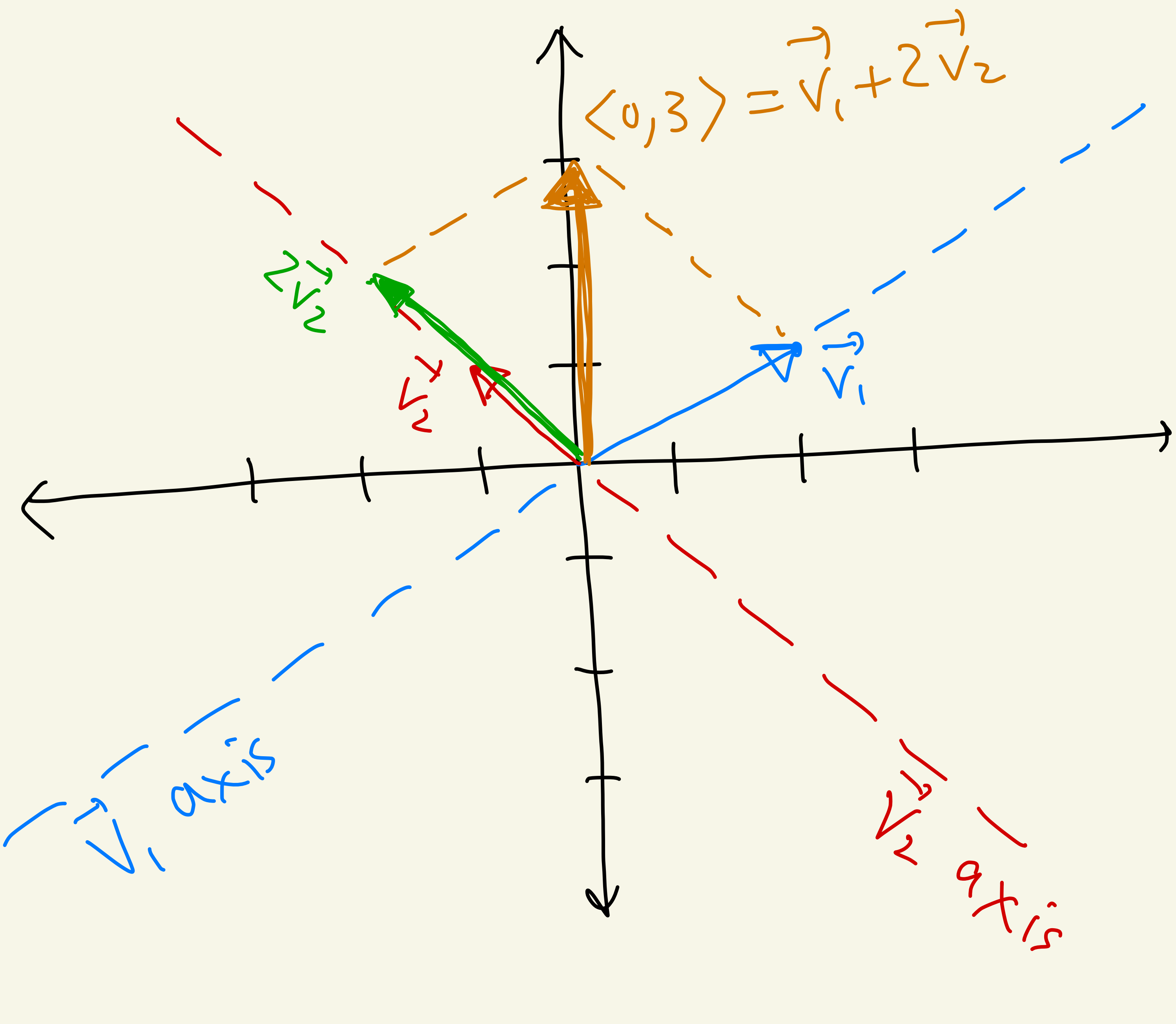
An example of a vector in the span of $\vec{v}_1, \vec{v}_2$ is

$$1 \cdot \vec{v}_1 + 2 \cdot \vec{v}_2 = 1 \cdot \langle 2, 1 \rangle + 2 \cdot \langle -1, 1 \rangle$$

$$= \langle 2, 1 \rangle + \langle -2, 2 \rangle$$

$$= \langle 0, 3 \rangle$$

So, $\langle 0, 3 \rangle$ is in the span of $\vec{v}_1, \vec{v}_2$.



$$\boxed{\text{\underline{Claim}: } \vec{v}_1 = \langle 2,1\rangle,\ \vec{v}_2 = \langle -1,1\rangle \text{ span all of } V = \mathbb{R}^2.}$$

__Proof:__ Let $\langle a,b\rangle$ be any vector in $\mathbb{R}^2$. We must show that can write

$$\langle a,b\rangle = \underbrace{c_1\langle 2,1\rangle + c_2\langle -1,1\rangle}_{c_1\vec{v}_1 + c_2\vec{v}_2}$$

This becomes

$$\langle a,b\rangle = \langle 2c_1, c_1\rangle + \langle -c_2, c_2\rangle$$

which gives

$$\langle a,b\rangle = \langle 2c_1 - c_2,\ c_1 + c_2\rangle$$

This gives

$$\boxed{\begin{aligned} 2c_1 - c_2 &= a \\ c_1 + c_2 &= b \end{aligned}}$$

Let's solve:

$$\left(\begin{array}{cc|c} 2 & -1 & a \\ 1 & 1 & b \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 2 & -1 & a \end{array}\right)$$

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 0 & -3 & a-2b \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 0 & 1 & -\frac{1}{3}a + \frac{2}{3}b \end{array}\right)$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 &= b \quad (1) \\ c_2 &= -\frac{1}{3}a + \frac{2}{3}b \quad (2) \end{aligned}}$$

② $c_2 = -\frac{1}{3}a + \frac{2}{3}b$

① $c_1 = b - c_2 = b - \left(-\frac{1}{3}a + \frac{2}{3}b\right)$

$$= \frac{1}{3}a + \frac{1}{3}b$$

Plugging this back into

$$\langle a, b \rangle = c_1 \langle 2, 1 \rangle + c_2 \langle -1, 1 \rangle$$

gives

$$\langle a, b \rangle = \underbrace{\left(\frac{1}{3}a + \frac{1}{3}b\right)\langle 2, 1 \rangle + \left(-\frac{1}{3}a + \frac{2}{3}b\right)\langle -1, 1 \rangle}_{\left(\frac{1}{3}a + \frac{1}{3}b\right)\vec{v}_1 + \left(-\frac{1}{3}a + \frac{2}{3}b\right)\vec{v}_2}$$

So, every vector $\langle a, b \rangle$ is in the span of $\vec{v}_1 = \langle 2, 1 \rangle$, $\vec{v}_2 = \langle -1, 1 \rangle$

$\boxed{\text{Claim}}$

<u>Ex:</u> $\langle a,b\rangle = \langle 3,-6\rangle$

$$\langle 3,-6\rangle = \left(\tfrac{1}{3}\cdot 3+\tfrac{1}{3}(-6)\right)\langle 2,1\rangle + \left(-\tfrac{1}{3}\cdot 3+\tfrac{2}{3}(-6)\right)\langle -1,1\rangle$$

Ie

$$\langle 3,-6\rangle = (-1)\langle 2,1\rangle + (-5)\langle -1,1\rangle$$

<u>Theorem:</u> Let V be a vector space over a field F. Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be vectors from V. Then $\text{span}(\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\})$ is a subspace of V.

picture when $n=3$

V

$\mathrm{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$

$\vec{v}_1 = 1\cdot\vec{v}_1 + 0\cdot\vec{v}_2 + 0\cdot\vec{v}_3$

$\vec{v}_2 = 0\cdot\vec{v}_1 + 1\cdot\vec{v}_2 + 0\cdot\vec{v}_3$

$\vec{0} = 0\cdot\vec{v}_1 + 0\cdot\vec{v}_2 + 0\cdot\vec{v}_3$

$100\vec{v}_1 + \frac{1}{2}\vec{v}_2 - 5000\vec{v}_3$

$\vec{v}_1 - \vec{v}_2$