Math 2550-01

3/19/24

Ex: Let $F = \mathbb{R}$ ← scalars

and

vectors

$$V = P_2 = \{ a + bx + cx^2 \mid a, b, c \in \mathbb{R} \}$$

up to degree 2 polynomials

$0 + 0x + 1 \cdot x^2$

$$= \{ 1 - 2x + x^2,\ 5,\ x^2,\ 1 - x + \pi x^2, \dots \}$$

$5 + 0x + 0x^2$

ex of adding two "vectors":

$$(1 - 2x + x^2) + (5 + 10x - 2x^2)$$

$$= 6 + 8x - x^2$$

ex of scaling a vector:

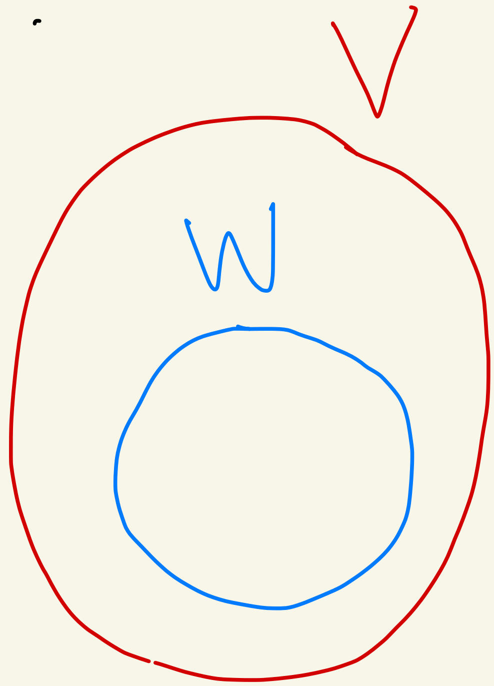
$$5 \cdot (1 - 2x + x^2) = 5 - 10x + 5x^2$$

$V = P_2$ is a vector space over $F = \mathbb{R}$

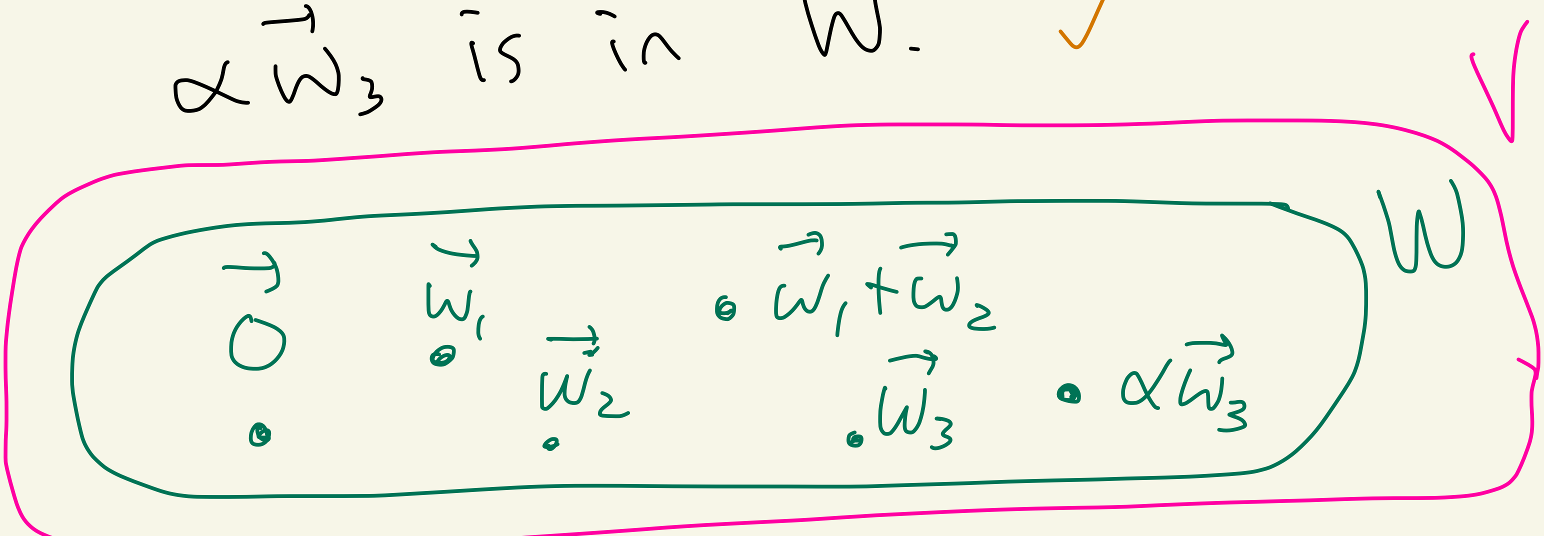
field	vectors
$\mathbb{R}$	$V = P_2$
0, 1, π, -2, $\sqrt{2}$	$1-2x+x^2$, x^2, 5, $\vec{0} = 0+0x+0x^2$

Now we define a subspace which is a vector space inside a vector space.

Def: Let V be a vector space over a field F. Let W be a subset of V. We say that W is a <u>subspace</u> of V if



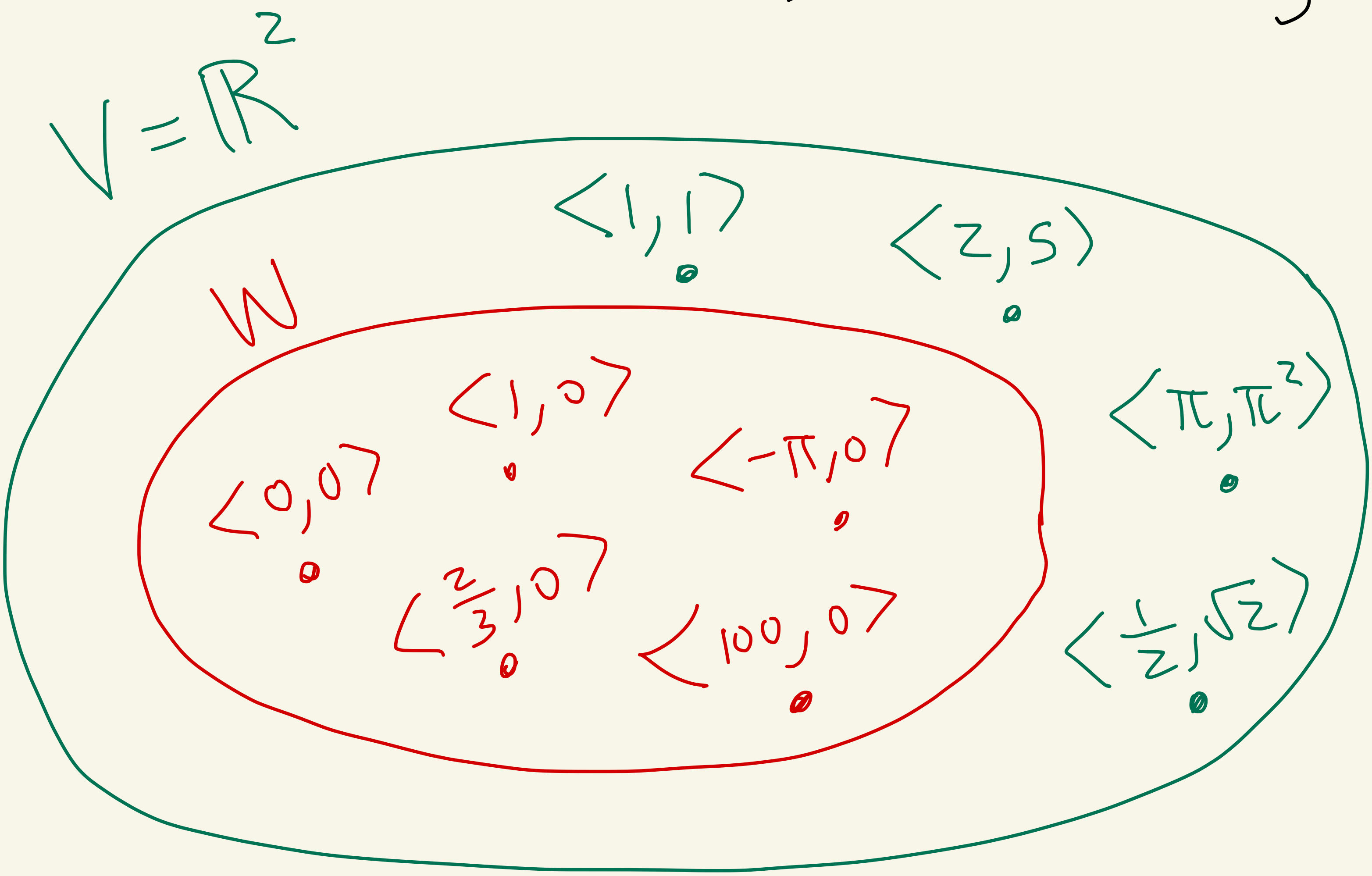
1. $\vec{0}$ is in W
2. if $\vec{w}_1, \vec{w}_2$ are in W, then $\vec{w}_1 + \vec{w}_2$ is in W. — W is closed under adding
3. if $\vec{w}_3$ is in W and α is in F, then $\alpha \vec{w}_3$ is in W. — W is closed under scaling

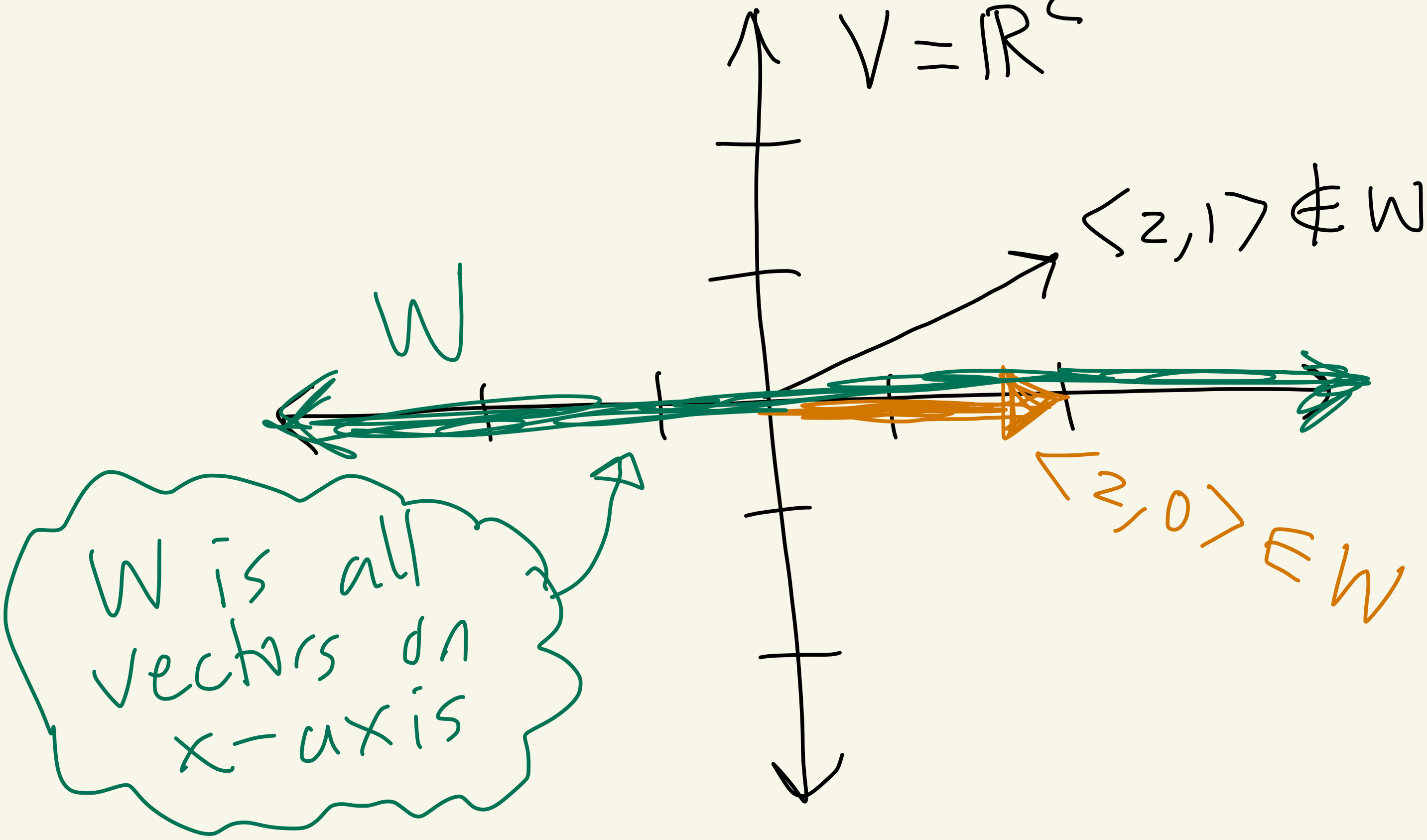


Ex: Let $F = \mathbb{R}$ and $V = \mathbb{R}^2$. This is a vector space

Let

$$W = \{ \langle x, 0 \rangle \mid x \in \mathbb{R} \}$$
$$= \{ \langle 1, 0 \rangle, \langle \tfrac{2}{3}, 0 \rangle, \langle -\pi, 0 \rangle, \langle 0, 0 \rangle, \langle 2, 0 \rangle, \ldots \}$$





Proof that W is a subspace:

① $\vec{0} = \langle 0,0 \rangle$ is in W ← set $x=0$ in $\langle x,0 \rangle$

② Let $\vec{w}_1$ and $\vec{w}_2$ be in W.

Then, $\vec{w}_1 = \langle x_1, 0 \rangle$

and $\vec{w}_2 = \langle x_2, 0 \rangle$

where x_1, x_2 are real numbers.

Then, $\vec{w}_1 + \vec{w}_2 = \langle x_1 + x_2, 0 \rangle$

is of the form $\langle x, 0 \rangle$

and so $\vec{w}_1 + \vec{w}_2$ is in W.

(3) Let $\vec{w}_3$ be in W and

α be a scalar in $F = \mathbb{R}$.

Then, $\vec{w}_3 = \langle x_3, 0 \rangle$

where $x_3 \in \mathbb{R}$.

And, $\alpha \vec{w}_3 = \alpha \langle x_3, 0 \rangle$

$$= \langle \alpha x_3, \alpha 0 \rangle$$

$$= \langle \alpha x_3, 0 \rangle$$

We see that $\alpha \vec{w}_3$ is of

the form $\langle x, 0 \rangle$ and

thus $\alpha \vec{w}_3$ is in W.

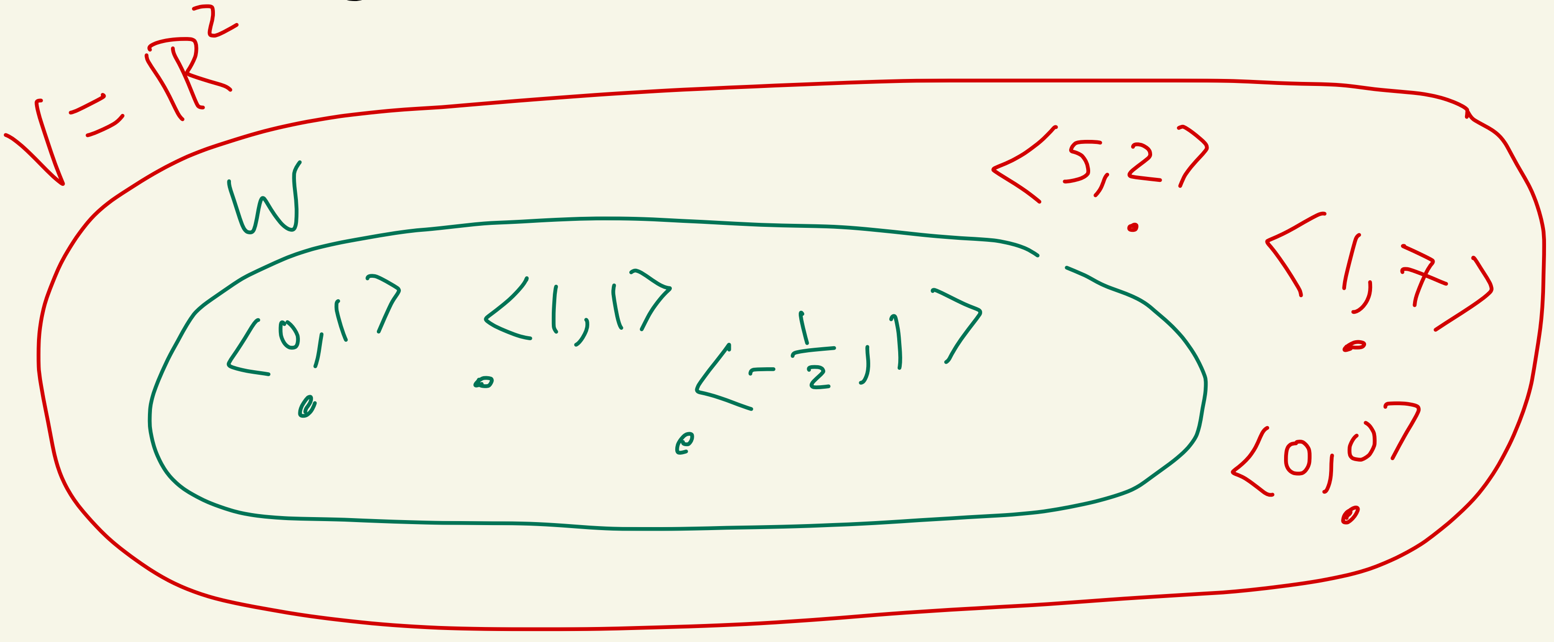
By ①, ②, ③, W is a subspace of V. $\blacksquare$

Ex: Let $F = \mathbb{R}$ and $V = \mathbb{R}^2$.

Let

$$W = \{ \langle x, 1 \rangle \mid x \in \mathbb{R} \}$$

$$= \{ \langle 0,1 \rangle, \langle 1,1 \rangle, \langle -\tfrac{1}{2}, 1 \rangle, \ldots \}$$



W is <u>not</u> a subspace.
To show this you just need to give an example of ①, ②, or ③ failing.
For example, ① isn't true since $\vec{0} = \langle 0,0 \rangle$ is not in W because it is not of the form $\langle x, 1 \rangle$.

You could instead say: W isn't a subspace because ② fails. For example, $\langle 2,1 \rangle$ and $\langle 4,1 \rangle$ are in W, but $\langle 2,1 \rangle + \langle 4,1 \rangle = \langle 6,2 \rangle$ is not in W.

Ex: Let $F = \mathbb{R}$ and

$$V = M_{2,2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R} \right\}$$

set of all 2x2 matrices

Let

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{array}{l} d = a+b \\ a,b,c,d \in \mathbb{R} \end{array} \right\}$$

$\vec{0}$

$$= \left\{ \underbrace{\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}}_{\substack{\text{is in } W \\ \text{since} \\ 2=1+1}}, \underbrace{\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}}_{\substack{\text{is in } W \\ \text{since} \\ 3=1+2}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\substack{\text{is in } W \\ \text{since} \\ 0=0+0}}, \ldots \right\}$$

Side calculations

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}}_{\text{in } W} + \underbrace{\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}}_{\text{in } W} = \underbrace{\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}}_{\substack{\text{in } W \\ \text{since } 2+3=5}}$$

$$5 \cdot \underbrace{\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}}_{\text{in } W} = \underbrace{\begin{pmatrix} 5 & 5 \\ 15 & 10 \end{pmatrix}}_{\substack{\text{in } W \\ \text{since } 5+5=10}}$$

It looks like its gonna be a subspace. Let's try to prove it.

Proof that W is a subspace

① Set $a=0, b=0, c=0, d=0$ to get $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and notice that $d = a+b$.

So, $\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in W.

② Let $\vec{w}_1$ and $\vec{w}_2$ be in W.

Then, $\vec{w}_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ where $\boxed{d_1 = a_1 + b_1}$

and $\vec{w}_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ where $\boxed{d_2 = a_2 + b_2}$

Then,

$$\vec{w}_1 + \vec{w}_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$ and

$$d_1 + d_2 = (a_1 + b_1) + (a_2 + b_2)$$

$$d_1 = a_1 + b_1$$
$$d_2 = a_2 + b_2$$

$$= (a_1 + a_2) + (b_1 + b_2).$$

So, $\vec{w}_1 + \vec{w}_2$ is in W still.

③ Let $\vec{w}_3$ be in W and α be a scalar.

Then, $\vec{w}_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$ where $d_3 = a_3 + b_3$.

We get

$$\alpha \vec{w}_3 = \begin{pmatrix} \alpha a_3 & \alpha b_3 \\ \alpha c_3 & \alpha d_3 \end{pmatrix}$$

and $\alpha d_3 = \alpha(a_3 + b_3) = \alpha a_3 + \alpha b_3$

$$d_3 = a_3 + b_3$$

Thus, $\alpha\vec{w}_3$ is in W.

By 1, 2, 3, W is a subspace.