Math 2550-01
3/14/24

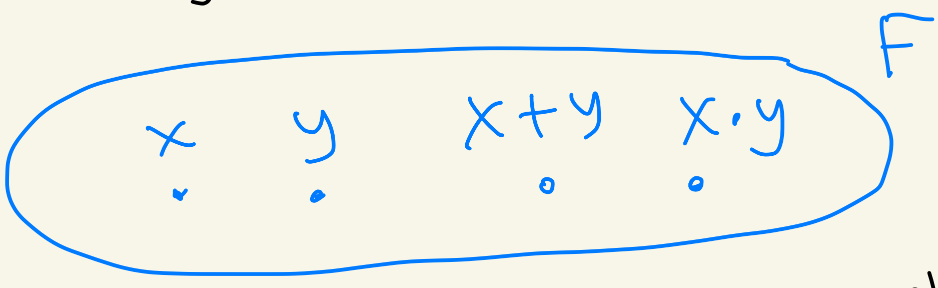
Topic 6 – Vector Spaces

field

We are going to generalize what a number/scalar is and what a vector is.

vector space

Def: A **field** consists of a set F of "scalars/numbers" and two operations $+$ and $\cdot$ such that if x and y are in F then $x+y$ and $x \cdot y$ are in F.



Also the following properties must hold:

(F1) If a, b, c are in F, then

$$a+b=b+a$$
$$a \cdot b = b \cdot a$$
$$a+(b+c)=(a+b)+c$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$
$$a \cdot (b+c) = a \cdot b + a \cdot c$$
$$(b+c) \cdot a = b \cdot a + c \cdot a$$

(F2) There exist unique elements 0 and 1 in F where

$$x+0=x+0=x$$
$$x \cdot 1 = 1 \cdot x = x$$

for all x in F.

(F3) Let x be in F.

There exists a unique element $-x$ in F where $x+(-x)=0=(-x)+x$

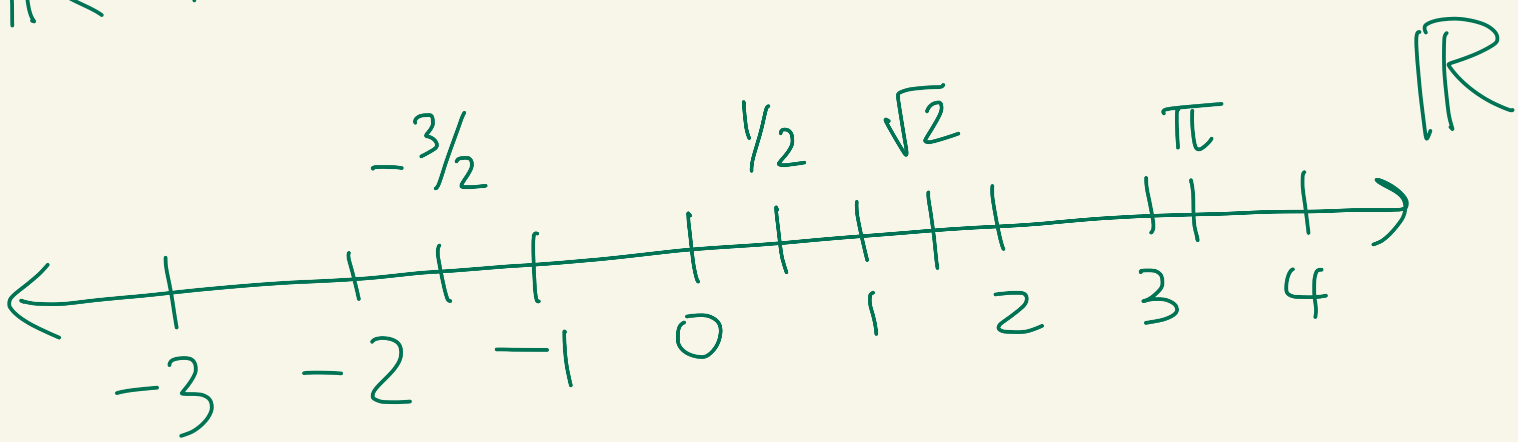
If $x \neq 0$, then there exists a unique element x^{-1} in F where

$$x \cdot x^{-1} = 1 = x^{-1} \cdot x$$

END OF DEF

<u>Ex:</u> $F = \mathbb{R}$ ← the set of real numbers

$\mathbb{R}$ is a field

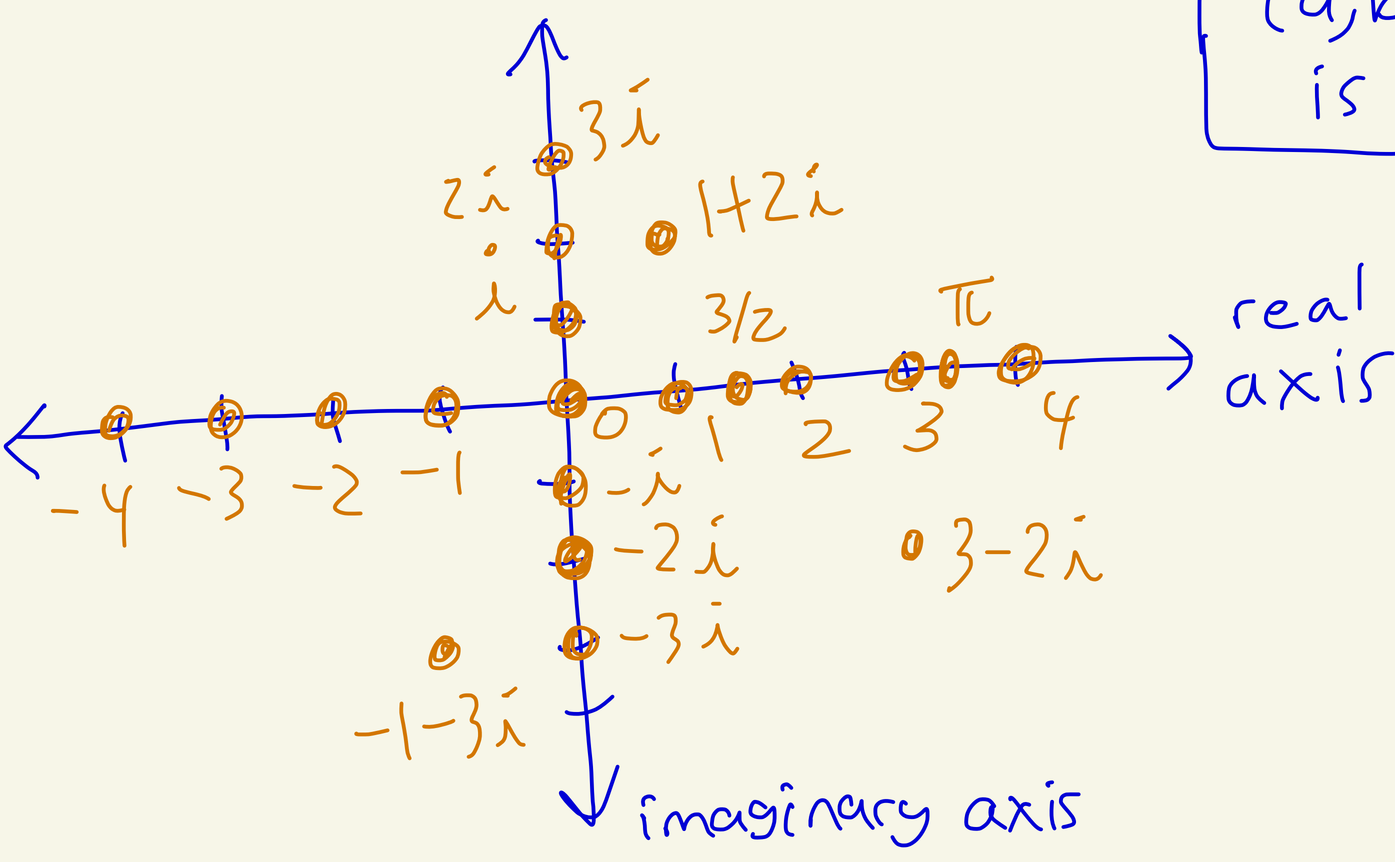


$\mathbb{R}$ is the only example we will use in this class, but I will state definitions and theorems for general fields. Just think $\mathbb{R}$ when you see the word "field".

$i^2 = -1$, $i = \sqrt{-1}$

<u>Ex:</u> The set of complex numbers $\mathbb{C}$ is a field.

plot $a+bi$ where (a,b) is



Ex: There are also finite fields. These come from prime numbers modular arithmetic.

Our next definition will generalize what a vector is.

<u>Def:</u> A <u>vector space</u> V over a field F consists of a set of "vectors" V and a field F with two operations, "vector addition" + and "vector scaling" $\cdot$, such that if $\vec{v}, \vec{w}, \vec{z}$ are vectors in V and α, β are scalars from F, then the following must hold:

1. $\vec{v} + \vec{w}$ is in V
2. $\alpha \cdot \vec{v}$ is in V

adding vectors and scaling a vector is still a vector

3. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
4. $\vec{v} + (\vec{w} + \vec{z}) = (\vec{v} + \vec{w}) + \vec{z}$

(5) there exists a unique vector $\vec{0}$ in V, called the ~~zero~~ vector, where $\vec{0} + \vec{y} = \vec{y} + \vec{0} = \vec{y}$ for all $\vec{y}$ in V.

(6) there exists a vector $-\vec{v}$ in V where $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$

(7) $1 \cdot \vec{v} = \vec{v}$ ← scaling by 1 does nothing

(8) $(\alpha\beta) \cdot \vec{v} = \alpha \cdot (\beta \cdot \vec{v})$

(9) $\alpha \cdot (\vec{v} + \vec{w}) = \alpha \cdot \vec{v} + \alpha \cdot \vec{w}$

(10) $(\alpha + \beta) \cdot \vec{v} = \alpha \cdot \vec{v} + \beta \vec{v}$

$\boxed{\text{END OF DEF}}$

Ex: Let $V = \mathbb{R}^n$ and $F = \mathbb{R}$

Using the usual vector adding and scaling makes this a vector space.

For example, when $n=2$ we have:

field	vectors
$F = \mathbb{R}$	$V = \mathbb{R}^2$
π, 1, 0, 2, 10, $1/2$	$\langle 0,0 \rangle = \vec{0}$, $\langle 5, \frac{1}{2} \rangle$, $\langle 3, -10 \rangle$

adding vectors:

$$\langle 1, 2 \rangle + \langle -3, 5 \rangle = \langle -2, 7 \rangle$$

scaling a vector:

$$5 \cdot \langle 2, 1 \rangle = \langle 10, 5 \rangle$$

__Ex:__ Let

$V = M_{2,2}$ be the set of all 2×2 matrices with real number entries and $F = \mathbb{R}$.

In set notation:

$$V = M_{2,2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ \pi & 2 \end{pmatrix}, \begin{pmatrix} 1/2 & 2 \\ 3 & 5 \end{pmatrix}, \ldots \right\}$$

$\boxed{\text{zero vector } \vec{0}}$

infinitely many

Adding and scaling is like normal, ie like this:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

FIELD	VECTORS
$F = \mathbb{R}$	$V = M_{2,2}$
$\frac{1}{2}$, 1, 2, 0, -1, π	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \vec{0}$, $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$, $\begin{pmatrix} 5 & 4 \\ -1 & \frac{1}{2} \end{pmatrix}$

Ex: Pick some integer $n \geq 0$.
(So n can be $0, 1, 2, 3, 4, 5, \ldots$)
Let $V = P_n$ be the set of all polynomials of degree $\leq n$.

So,

vectors

$$V = P_n = \left\{ a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \;\middle|\; \begin{array}{c} a_0, a_1, \ldots, a_n \\ \text{are real} \\ \text{numbers} \end{array} \right\}$$

Let $F = \mathbb{R}$. $\leftarrow$ scalars

Add and scale vectors like usual.
For example,

$$(a_0 + a_1 x + \ldots + a_n x^n) + (b_0 + b_1 x + \ldots + b_n x^n)$$
$$= (a_0 + b_0) + (a_1 + b_1) x + \ldots + (a_n + b_n) x^n$$

and

$$\alpha(a_0 + a_1 x + \dots + a_n x^n)$$
$$= (\alpha a_0) + (\alpha a_1) x + \dots + (\alpha a_n) x^n$$

Also,

$$a_0 + a_1 x + \dots + a_n x^n = b_0 + b_1 x + \dots + b_n x^n$$

if and only if

$$a_0 = b_0,\ a_1 = b_1,\ \dots,\ a_n = b_n$$

And,

$$\vec{0} = 0 + 0x + 0x^2 + \dots + 0x^n$$

This will be a vector space.