Math 2550-01
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Ex:

$$\begin{pmatrix} 1 & -1 \\ 5 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 1+0 & -1+2 \\ 5+1 & 3-5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 6 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 \\ 3 & 1 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ -5 & -2 \end{pmatrix} = \begin{pmatrix} 5-1 & 0-2 \\ 3-3 & 1-4 \\ 1-(-5) & -1-(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 \\ 0 & -3 \\ 6 & 1 \end{pmatrix}$$

$$(-2)\begin{pmatrix} 3 & 2 \\ 0 & -1 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} (-2)(3) & (-2)(2) \\ (-2)(0) & (-2)(-1) \\ (-2)(5) & (-2)(-3) \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ 0 & 2 \\ -10 & 6 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}}_{2\times 2} + \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 5 & 3 \end{pmatrix}}_{3\times 2}$$

is undefined since the matrices don't have the same size

To be able to add or subtract two matrices, both matrices have to be exactly the same size. Otherwise, the operation is undefined.

Def: Let A be an $m \times r$ matrix and let B be an $r \times n$ matrix. We define the *product* of A and B, denoted by AB, as the $m \times n$ matrix C whose entry at row i and column j is defined to be the dot product of row i of A with column j of B.

$$\underbrace{C}_{m \times n} = \underbrace{A}_{m \times r} \; \underbrace{B}_{r \times n}$$

must equal

Ex: Calculate AB, if possible, where

$$A=\underbrace{\begin{pmatrix}1&2\\-1&0\end{pmatrix}}_{2\times 2} \quad \text{and} \quad B=\underbrace{\begin{pmatrix}1&2&-1\\0&1&0\end{pmatrix}}_{2\times 3}$$

equal

AB will be 2×3

$$AB=\begin{pmatrix}\overbrace{\begin{pmatrix}1&2\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}}^{(\text{row } 1\ A)\cdot(\text{col } 1\ B)} & \overbrace{\begin{pmatrix}1&2\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix}}^{(\text{row } 1\ A)\cdot(\text{col } 2\ B)} & \overbrace{\begin{pmatrix}1&2\end{pmatrix}\cdot\begin{pmatrix}-1\\0\end{pmatrix}}^{(\text{row } 1\ A)\cdot(\text{col } 3\ B)} \\ \\ \underbrace{\begin{pmatrix}-1&0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}}_{(\text{row } 2\ A)\cdot(\text{col } 1\ B)} & \underbrace{\begin{pmatrix}-1&0\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix}}_{(\text{row } 2\ A)\cdot(\text{col } 2\ B)} & \underbrace{\begin{pmatrix}-1&0\end{pmatrix}\cdot\begin{pmatrix}-1\\0\end{pmatrix}}_{(\text{row } 2\ A)\cdot(\text{col } 3\ B)}\end{pmatrix}$$

$$= \begin{pmatrix} (1)(1)+(2)(0) & (1)(2)+(2)(1) & (1)(-1)+(2)(0) \\ (-1)(1)+(0)(0) & (-1)(2)+(0)(1) & (-1)(-1)+(0)(0) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 & -1 \\ -1 & -2 & 1 \end{pmatrix}$$

<u>Ex:</u> Using the same matrices

$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

can we calculate BA?

$$\underbrace{B}_{2\times 3}\ \underbrace{A}_{2\times 2}$$

$3 \neq 2$

since $3 \neq 2$
BA is undefined

You can also see this if you try to calculate it...

$$BA = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \overbrace{\boxed{\begin{pmatrix} 1 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}}}^{(\text{row } 1 \ B) \cdot (\text{col } 1 \ A)} & \sim\sim\sim \\ \sim\sim\sim & \sim\sim\sim \end{pmatrix}$$

this dot product can't be calculated since the sizes don't match

Ex: Let

$$A = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & -3 \end{pmatrix}$$

Calculate AB if possible

$$\underbrace{A}_{3\times 1}\ \underbrace{B}_{1\times 3}$$

same

AB is 3×3

$$AB = \begin{pmatrix} \overbrace{(1)\cdot(0)}^{(\text{row } 1\ A)\cdot(\text{col } 1\ B)} & \overbrace{(1)\cdot(1)}^{(\text{row } 1\ A)\cdot(\text{col } 2\ B)} & \overbrace{(1)\cdot(-3)}^{(\text{row } 1\ A)\cdot(\text{col } 3\ B)} \\ \overbrace{(2)\cdot(0)}^{(\text{row } 2\ A)\cdot(\text{col } 1\ B)} & \overbrace{(2)\cdot(1)}^{(\text{row } 2\ A)\cdot(\text{col } 2\ B)} & \overbrace{(2)\cdot(-3)}^{(\text{row } 2\ A)\cdot(\text{col } 3\ B)} \\ \overbrace{(-1)\cdot(0)}^{(\text{row } 3\ A)\cdot(\text{col } 1\ B)} & \overbrace{(-1)\cdot(1)}^{(\text{row } 3\ A)\cdot(\text{col } 2\ B)} & \overbrace{(-1)\cdot(-3)}^{(\text{row } 3\ A)\cdot(\text{col } 3\ B)} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -3 \\ 0 & 2 & -6 \\ 0 & -1 & 3 \end{pmatrix}$$

<u>Ex:</u> Let $A = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & -3 \end{pmatrix}$ as above. Can we calculate BA?

$$\underbrace{B}_{1\times 3}\ \underbrace{A}_{3\times 1} = \begin{pmatrix} 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

same

BA is 1x1

$$= \left(\underbrace{\begin{pmatrix} 0 & 1 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}}_{\substack{(\text{row } 1\ B)\ \cdot \\ (\text{col } 1\ A)}} \right)$$

$$= \left((0)(1) + (1)(2) + (-3)(-1) \right)$$

$$= (5)$$

BA is a 1×1 matrix

<u>Note:</u> We saw above that when $A = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & -3 \end{pmatrix}$ that $AB \neq BA$.

> In general, $AB = BA$ is <u>not</u> always true for matrices

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Def: Let A be an $m \times n$ matrix. The **transpose** of A, denoted by A^T, is defined to be the $n \times m$ matrix that results from interchanging the rows and columns of A.

That is, the i-th column of A^T is the i-th row of A.

Or you could say that the j-th row of A^T is the j-th column of A.

$\boxed{\text{Some people write } A^t \text{ instead of } A^T}$

Ex:

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix}$$

A is 3×2

$$A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{pmatrix}$$

A^T is 2×3

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$\boxed{A \text{ is } 3 \times 3}$

$$A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

$\boxed{A^T \text{ is } 3 \times 3}$

Def: The $m \times n$ **zero matrix** is the $m \times n$ matrix where every entry is zero.
We denote this matrix by $O_{m \times n}$ or just by O if we don't want to mention the size.

Ex: $O_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$O_{3\times 4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$O_{1\times 6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix}$

$$A + 0_{2\times2} = \begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix} = A$$

$$0_{2\times2} + A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix} = A$$

So, $\left.\begin{aligned} A + 0_{2\times2} &= A \\ 0_{2\times2} + A &= A \end{aligned}\right]$ So, $0_{2\times2}$ acts like the number 0 does