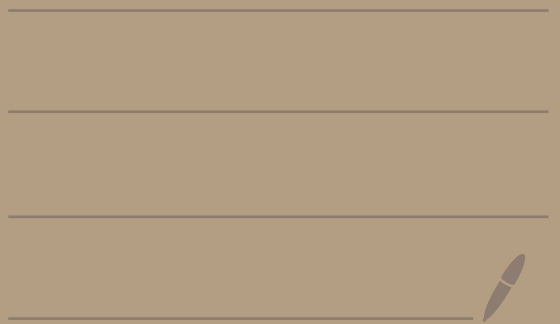


Math 2550-01

2/29/24



(topic 4 continued...)

Last time we talked about how to write a linear system in the form $A\vec{x} = \vec{b}$.

If A^{-1} exists then you can then solve for \vec{x} and you get this:

$$A\vec{x} = \vec{b}$$

$$\underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

So, if A^{-1} exists then there is one solution to the system.

Ex: Find all the solutions to

$$\begin{aligned} 3x &+ 3z = 9 \\ x + y + 2z &= -4 \\ -2x + 3y &= 5 \end{aligned}$$

(*)

Write the above system in the form $A\vec{x} = \vec{b}$.

$$A = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}, \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \vec{b} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

Check: $\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix} \Leftrightarrow A\vec{x} = \vec{b}$

$$\begin{pmatrix} 3x & + 3z \\ x + y + 2z \\ -2x + 3y \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

this is the same as (*)

So (*) is the same as

$$\underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{\vec{b}}$$

Previously, we found that A^{-1} exists

and $A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$.

Multiply by A^{-1} on the left side to get

$$\underbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}}_{I_3} \underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{\vec{b}}$$

$A^{-1}A\vec{x} = A^{-1}\vec{b}$

This becomes

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix} \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

This gives

$$\begin{pmatrix} x + 0y + 0z \\ 0x + y + 0z \\ 0x + 0y + z \end{pmatrix} = \begin{pmatrix} (2)(9) + (-3)(-4) + (1)(5) \\ (4/3)(9) + (-2)(-4) + (1)(5) \\ (-5/3)(9) + (3)(-4) + (-1)(5) \end{pmatrix}$$

So we get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 35 \\ 25 \\ -32 \end{pmatrix}$$

So (*) has exactly one solution

it is $x = 35, y = 25, z = -32$.

Topic 5 - Determinants

The determinant will allow us to detect when a square matrix has an inverse.

Def: Let A be an $n \times n$ matrix. The matrix $A_{\bar{i}\bar{j}}$ is defined to be the $(n-1) \times (n-1)$ matrix obtained by removing row i and column j from A .

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$$A_{32} = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = A$$

$$A_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = A$$

$$n! = \begin{cases} n \cdot (n-1)! & , n \geq 1 \\ 1 & , n = 0 \end{cases}$$

$$1! = 1 \cdot 0! = 1 \cdot 1 = 1$$

$$2! = 2 \cdot 1! = 2 \cdot 1 \cdot 0! = 2 \cdot 1 \cdot 1 = 2$$

Def: Let A be an $n \times n$ matrix.

Let a_{ij} be the entry in A at row i and column j .

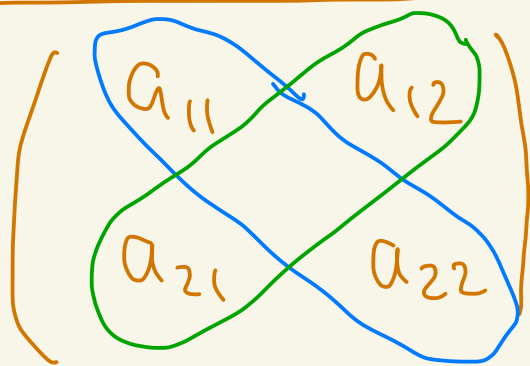
Define the determinant of A , denoted by $\det(A)$, as follows:

① If $n=1$ and $A = (a_{11})$,

then $\det(A) = a_{11}$.

② If $n=2$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

then $\det(A) = a_{11}a_{22} - a_{21}a_{12}$



③ If $n \geq 3$, then pick a column j to "expand on" and define

$$\det(A) = \sum_{\bar{i}=1}^n (-1)^{\bar{i}+\bar{j}} \cdot a_{\bar{i}\bar{j}} \cdot \det(A_{\bar{i}\bar{j}})$$

sum over rows i of A
column j is fixed

This is called the expansion
of the determinant along
the j -th column.

Note: In step 3, you can instead pick a row i and then

$$\det(A) = \sum_{\bar{j}=1}^n (-1)^{\bar{i}+\bar{j}} \cdot a_{\bar{i}\bar{j}} \cdot \det(A_{\bar{i}\bar{j}})$$

sum over columns j of A
row i is fixed

Note: It doesn't matter what row or column you pick in step 3. In the end, you'll always get the same answer.

Notation: We will also use bars instead of $\det(A)$ to notate the determinant.

Like this:

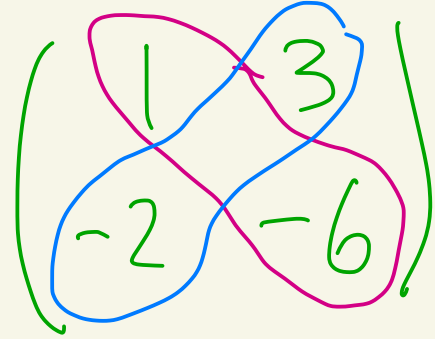
$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

Ex:

$$\det(7) = 7$$

Ex:

$$\det \begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix} = \underbrace{(1)(-6) - (3)(-2)}$$



$$= 0$$

Ex: $A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$

$$j=2$$



$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

Expand on column $j=2$

$$\det(A) = \sum_{\bar{i}=1}^3 (-1)^{\bar{i}+2} \cdot a_{\bar{i}2} \cdot \det(A_{\bar{i}2})$$

$$\bar{i}=1$$

Sum over rows
 $j=2$ is fixed

$$= (-1)^{1+2} \cdot a_{12} \cdot \det(A_{12})$$

← $\bar{i}=1$
term

$$+ (-1)^{2+2} \cdot a_{22} \cdot \det(A_{22})$$

← $\bar{i}=2$
term

$$+ (-1)^{3+2} \cdot a_{32} \cdot \det(A_{32})$$

← $\bar{i}=3$
term

$$= (-1)(1) \cdot \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} \quad \leftarrow \begin{pmatrix} \cancel{3} & \boxed{1} & \cancel{0} \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$+ (1)(-4) \cdot \begin{vmatrix} 3 & 0 \\ 5 & -2 \end{vmatrix} \quad \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ \cancel{-2} & \boxed{-4} & \cancel{3} \\ 5 & 4 & -2 \end{pmatrix}$$

$$+ (-1)(4) \cdot \begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix} \quad \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ \cancel{5} & \cancel{4} & \cancel{-2} \end{pmatrix}$$

$$= -((-2)(-2) - (3)(5))$$

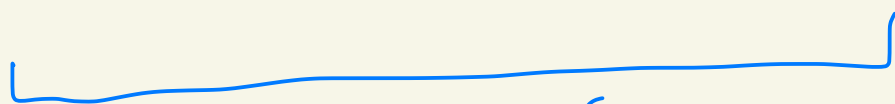
$$- 4((3)(-2) - (0)(5))$$

$$- 4((3)(3) - (0)(-2))$$

$$= \boxed{-1}$$

PICTURE WAY TO DETERMINE $(-1)^{\hat{i}+\hat{j}}$

$$\begin{pmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$



$$(-1)^{\hat{i}+\hat{j}}$$

\hat{i} is row

\hat{j} is column

↑
We did column 2