Math 2550-01
2/29/24

(topic 4 continued...)

Last time we talked about how to write a linear system in the form $A\vec{x} = \vec{b}$.

If A^{-1} exists then you can then solve for $\vec{x}$ and you get this:

$$A\vec{x} = \vec{b}$$

$$\underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

So, if A^{-1} exists then there is one solution to the system.

Ex: Find all the solutions to

$$\boxed{\begin{aligned} 3x \qquad\quad &+ 3z = 9 \\ x + y &+ 2z = -4 \\ -2x + 3y \qquad &= 5 \end{aligned}} \quad (*)$$

Write the above system in the form $A\vec{x} = \vec{b}$.

$$A = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

Check:

$$\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix} \leftarrow A\vec{x} = \vec{b}$$

$$\begin{pmatrix} 3x + 3z \\ x + y + 2z \\ -2x + 3y \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

this is the same as $(*)$

So (*) is the same as

$$\underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{\vec{b}}$$

Previously, we found that A^{-1} exists and $A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$.

Multiply by A^{-1} on the left side to get

$$\underbrace{\overbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}^{I_3} \begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{A^{-1}A\vec{x}} = \underbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix} \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{A^{-1}\vec{b}}$$

This becomes

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix} \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

This gives

$$\begin{pmatrix} x + 0y + 0z \\ 0x + y + 0z \\ 0x + 0y + z \end{pmatrix} = \begin{pmatrix} (2)(9) + (-3)(-4) + (1)(5) \\ \left(\frac{4}{3}\right)(9) + (-2)(-4) + (1)(5) \\ \left(-\frac{5}{3}\right)(9) + (3)(-4) + (-1)(5) \end{pmatrix}$$

So we get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 35 \\ 25 \\ -32 \end{pmatrix}$$

So $(*)$ has exactly one solution

it is $x = 35, y = 25, z = -32$.

Topic 5 - Determinants

The determinant will allow us to detect when a square matrix has an inverse.

<u>Def:</u> Let A be an $n \times n$ matrix. The matrix A_{ij} is defined to be the $(n-1)\times(n-1)$ matrix obtained by removing row i and column j from A.

<u>Ex:</u>

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A_{32} = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} \longleftarrow \boxed{\begin{pmatrix} 1 & \cancel{2} & 3 \\ 4 & \cancel{5} & 6 \\ \cancel{7} & \cancel{8} & \cancel{9} \end{pmatrix} = A}$$

$$A_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} \longleftarrow \boxed{\begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{4} & 5 & 6 \\ \cancel{7} & 8 & 9 \end{pmatrix} = A}$$

$$n! = \begin{cases} n \cdot ((n-1)!) & , \ n \geq 1 \\ 1 & , \ n = 0 \end{cases}$$

$$1! = 1 \cdot 0! = 1 \cdot 1 = 1$$

$$2! = 2 \cdot 1! = 2 \cdot 1 \cdot 0! = 2 \cdot 1 \cdot 1 = 2$$

<u>Def:</u> Let A be an $n \times n$ matrix.

Let a_{ij} be the entry in A at row i and column j.

Define the <u>determinant</u> of A, denoted by $\det(A)$, as follows:

1. If $n=1$ and $A = (a_{11})$, then $\det(A) = a_{11}$.

2. If $n=2$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\text{then } \underbrace{\det(A) = a_{11}a_{22} - a_{21}a_{12}}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

③ If $n \geq 3$, then pick a column j to "expand on" and define

$$\det(A) = \underbrace{\sum_{i=1}^{n}}_{\substack{\text{sum over rows } i \text{ of } A \\ \text{column } j \text{ is fixed}}} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

This is called the <u>expansion of the determinant along the j-th column.</u>

<u>Note:</u> In step 3, you can instead pick a row i and then

$$\det(A) = \underbrace{\sum_{j=1}^{n}}_{\substack{\text{sum over columns } j \text{ of } A \\ \text{row } i \text{ is fixed}}} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

Note: It doesn't matter what row or column you pick in step 3. In the end, you'll always get the same answer.

Notation: We will also use bars instead of det(A) to notate the determinant.

Like this:

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

Ex:

$$\det\begin{pmatrix} 7 \end{pmatrix} = 7$$

Ex:

$$\det\begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix} = \underbrace{(1)(-6) - (3)(-2)}_{\begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix}}$$

$$= 0$$

Ex:

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$\boxed{j=2}$

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

Expand on column $j=2$

$$\det(A) = \underbrace{\sum_{i=1}^{3}}_{\text{sum over rows}} (-1)^{i+2} \cdot a_{i2} \cdot \det(A_{i2})$$

sum over rows
$j=2$ is fixed

$$= (-1)^{1+2} \cdot a_{12} \cdot \det(A_{12})$$

$i=1$ term

$$+ (-1)^{2+2} \cdot a_{22} \cdot \det(A_{22})$$

$i=2$ term

$$+ (-1)^{3+2} \cdot a_{32} \cdot \det(A_{32})$$

$i=3$ term

$$= (-1)(1)\cdot\begin{vmatrix}-2 & 3\\ 5 & -2\end{vmatrix} \quad \longleftarrow \quad \begin{pmatrix}3 & 1 & 0\\ -2 & -4 & 3\\ 5 & 4 & -2\end{pmatrix}$$

$$+ (1)(-4)\cdot\begin{vmatrix}3 & 0\\ 5 & -2\end{vmatrix} \quad \longleftarrow \quad \begin{pmatrix}3 & 1 & 0\\ -2 & -4 & 3\\ 5 & 4 & -2\end{pmatrix}$$

$$+ (-1)(4)\cdot\begin{vmatrix}3 & 0\\ -2 & 3\end{vmatrix} \quad \longleftarrow \quad \begin{pmatrix}3 & 1 & 0\\ -2 & -4 & 3\\ 5 & 4 & -2\end{pmatrix}$$

$$= -\big((-2)(-2)-(3)(5)\big)$$

$$-4\big((3)(-2)-(0)(5)\big)$$

$$-4\big((3)(3)-(0)(-2)\big)$$

$$= -1$$

PICTURE WAY TO DETERMINE $(-1)^{i+j}$

$$\underbrace{\begin{pmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{pmatrix}}_{(-1)^{i+j}} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

i is row
j is column

We did column 2