Math 2550-01

2/27/24

<u>Theorem:</u> Let A be an $n \times n$ matrix that is invertible, then there exists only <u>one</u> $n \times n$ matrix B that is an inverse of A

[ie where $AB = BA = I_n$]

<u>Notation:</u> If A is invertible, then we can denote its unique inverse by $A^{-1} \leftarrow \boxed{B}$

<u>Ex:</u> $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

Last time we saw that $A^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$

How to find A^{-1} if it exists

Let A be an $n \times n$ matrix ← square matrix

<u>Procedure:</u> Start with the matrix

$$\left(A \,\middle|\, I_n \right)$$

Do row reduction on the above matrix until the left side is either I_n or has a row of zeros.

If you end up with I_n on the left side, then the right side will have A^{-1} in it.

<u>If</u> you end up with a row of zeros on the left side then A^{-1} does not exist.

Ex: Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$. ← $\boxed{2\times 2}$

Find A^{-1} if it exists.

Goal: Try to get I_2 on the left side

$$\left(\underbrace{\begin{matrix} 1 & 1 \\ 2 & 1 \end{matrix}}_{A} \;\middle|\; \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2}\right) \xrightarrow{-2R_1+R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & \boxed{-1} & -2 & 1 \end{array}\right)$$

make this 1

make 0

$$\xrightarrow{-R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & \boxed{1} & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

$$\xrightarrow{-R_2+R_1 \to R_1} \left(\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} \;\middle|\; \underbrace{\begin{matrix} -1 & 1 \\ 2 & -1 \end{matrix}}_{A^{-1}}\right)$$

So, $A^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$

Ex: Find A^{-1} if it exists when $A = \begin{pmatrix} 1 & 5 \\ -2 & -10 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} \overbrace{\begin{matrix} 1 & 5 \\ \boxed{-2} & -10 \end{matrix}}^{A} & & \overbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}^{I_2} & \end{array}\right) \xrightarrow{2R_1 + R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right)$$

make 0

row of zeros on left side.

So, $A = \begin{pmatrix} 1 & 5 \\ -2 & -10 \end{pmatrix}$ has no inverse

There is no 2×2 matrix B where $AB = BA = I_2$.

Ex: Find A^{-1} if it exists

where $A = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}$ ← $\boxed{3 \times 3}$

$$\left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{A} & \multicolumn{3}{c}{I_3} \\ \boxed{3} & 0 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

need a 1 here

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 3 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

make these 0

$$\xrightarrow[2R_1 + R_3 \to R_3]{-3R_1 + R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & \boxed{-3} & -3 & 1 & -3 & 0 \\ 0 & 5 & 4 & 0 & 2 & 1 \end{array}\right)$$

make this 1

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1/3 & 1 & 0 \\ 0 & \boxed{5} & 4 & 0 & 2 & 1 \end{array}\right)$$

make these 0

$$\xrightarrow[-5R_2+R_3 \to R_3]{-R_2+R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/3 & 0 & 0 \\ 0 & 1 & 1 & -1/3 & 1 & 0 \\ 0 & 0 & -1 & 5/3 & -3 & 1 \end{array}\right)$$

make this 1

$$\xrightarrow{-R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \boxed{1} & 1/3 & 0 & 0 \\ 0 & 1 & \boxed{1} & -1/3 & 1 & 0 \\ 0 & 0 & 1 & -5/3 & 3 & -1 \end{array}\right)$$

make these 0

$$\xrightarrow[-R_3+R_2 \to R_2]{-R_3+R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 1 \\ 0 & 1 & 0 & 4/3 & -2 & 1 \\ 0 & 0 & 1 & -5/3 & 3 & -1 \end{array}\right)$$

$$\underbrace{\quad\quad}_{I_3} \qquad \underbrace{\quad\quad}_{A^{-1}}$$

So when $A = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}$

we have $A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$.

This means

$$AA^{-1} = \underset{\uparrow \atop \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}{I_3} = A^{-1}A$$

Theorem Let A and B be $n \times n$ matrices that are both invertible.

[Ie, A^{-1} and B^{-1} exist.]

Then:

① AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

note: $(AB)^{-1} \neq A^{-1}B^{-1}$

② A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Let's use inverses to solve systems. We need a new way to write a system of linear equations.

Given the system

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}} \quad (*)$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Then the system $(*)$ is equivalent to

$$\underbrace{A\vec{x}}_{\text{matrix multiplication}} = \vec{b}$$

Ex: Consider the system

$$\boxed{\begin{aligned} x + 2y &= -1 \\ 3x - 5y &= 7 \end{aligned}} \quad (*)$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

Then

$$A\vec{x} = \vec{b}$$

becomes

$$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

which becomes

$$\begin{pmatrix} (1 \ \ 2) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ (3 \ -5) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

which gives

$$\begin{pmatrix} x+2y \\ 3x-5y \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

Which is equivalent to $(*)$

$$x+2y=-1$$
$$3x-5y=7$$